Infinitary cut-elimination via finite approximations

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Abstract

We investigate non-wellfounded proof systems based on parsimonious logic, a weaker variant of linear logic where the exponential modality ! is interpreted as a constructor for streams over finite data. Logical consistency is maintained at a global level by adapting a standard progressing criterion. We present an infinitary version of cut-elimination based on finite approximations, and we prove that, in presence of the progressing criterion, it returns well-defined non-wellfounded proofs at its limit. Furthermore, we show that cut-elimination preserves the progressive criterion and various regularity conditions internalizing degrees of proof-theoretical uniformity. Finally, we provide a denotational semantics for our systems based on the relational model.

2012 ACM Subject Classification Theory of computation → Linear logic; Theory of computation → Proof theory

Keywords and phrases cut-elimination, non-wellfounded proofs, parsimonious logic, linear logic, proof theory, approximation, sequent calculus, non-uniform proofs

Digital Object Identifier 10.4230/LIPIcs.CVIT.2016.23

1 Introduction

Non-wellfounded proof theory studies proofs as possibly infinite (but finitely branching) trees, where logical consistency is maintained via global conditions called *progressing* (or *validity*) *criteria*. In this setting, the so-called *regular* (also called *circular*) proofs receive a special attention, as they admit a finite description in terms of (possibly cyclic) directed graphs.

This area of proof theory makes its first appearance (in its modern guise) in the modal μ-calculus [25, 11]. Since then, it has been extensively investigated from many perspectives (see, e.g., [5, 30, 10, 19]), establishing itself as an ideal setting for manipulating least and greatest fixed points, and hence for modeling induction and coinduction principles.

Non-wellfounded proof theory has been applied to constructive fixed point logics i.e., with a computational interpretation based on the *Curry-Howard correspondence* [31]. A key example can be found in the context of *linear logic* (LL) [17], a logic implementing a finer control on resources thanks to the *exponential* modalities ! and ?. In this framework, the most extensively studied fixed point logic is μMALL, defined as the exponential-free fragment of LL with least and greatest fixed point operators (respectively, μ and its dual ν) [4, 3].

In [4] Baelde and Miller have shown that the exponentials can be recovered in μMALL by exploiting the fixed points operators, i.e., by defining $!A := \nu X.(\mathbf{1} \,\&\, A \,\&\, (X \otimes X))$ and $?A := \mu X.(\bot \oplus A \oplus (X \parr X))$. As these authors notice, the fixed point-based definition of ! and ? can be regarded as a more permissive variant of the standard exponentials, since a proof of $\nu X.(\mathbf{1} \,\&\, A \,\&\, (X \otimes X))$ could be constructed using different proofs of A, whereas in LL a proof of $!A$ is constructed uniformly using a single proof of A. This proof-theoretical notion of *non-uniformity* is indeed a central feature of the fixed-point exponentials.

However, the above encoding is not free from issues. First, as discussed in full detail in [13], the encoding of the exponentials does not verify the Seely isomorphisms, syntactically expressed by the equivalence $!(A \,\&\, B) \multimapboth (!A \otimes !B)$, an essential property for modeling exponentials in LL. Specifically, the fixed-point definition of ! relies on the multiplicative connective $\otimes$, which forces an interpretation of $!A$ based on lists rather than multisets. Secondly, as pointed out in [4], there is a neat mismatch between cut-elimination for the exponentials of LL and the one for the fixed point exponentials of μMALL. While the first problem is related to syntactic deficiencies of the encoding, and does not undermine further investigations on fixed point-based definitions of the exponential modalities, the second one is more critical. These apparent differences between the two exponentials contribute to stressing an important aspect in linear logic modalities, i.e., their *non-canonicity* [27, 9][1].

On a parallel research thread, Mazza [22, 23] studied *parsimonious logic*, a variant of linear logic where the exponential modality ! satisfies Milner's law (i.e., $!A \multimapboth A \otimes !A$) and invalidates the implications $!A \multimap !!A$ (*digging*) and $!A \multimap !A \otimes !A$ (*contraction*). In parsimonious logic, a proofs of $!A$ can be interpreted as a *stream* over (a finite set of) proofs of A, i.e., as a greatest fixed point, where the linear implications $A \otimes !A \multimap !A$ (*co-absorption*) and $!A \multimap A \otimes !A$ (*absorption*) can be computationally read as the *push* and *pop* operations on streams. More specifically, a formula $!A$ is introduced by an *infinitely branching rule* that takes a finite set of proofs $\mathcal{D}_1, \ldots, \mathcal{D}_n$ of A and a (possibly non-recursive) function $f : \mathbb{N} \to \{1, \ldots, n\}$ as premises, and constructs a proof of $!A$ representing a stream of proofs of the form $\mathfrak{S} = (\mathcal{D}_{f(0)}, \mathcal{D}_{f(1)}, \ldots, \mathcal{D}_{f(n)}, \ldots)$. Hence, parsimonious logic exponential modalities exploit in an essential way the above-mentioned proof-theoretical non-uniformity, which in turn deeply interfaces with notions of non-uniformity from computational complexity [23].

The analysis of parsimonious logic conducted in [22, 23] reveals that fixed point definitions of the exponentials are better behaving when digging and contraction are discarded. On the other hand, the co-absorption rule cannot be derived in LL, and so it prevents parsimonious logic becoming a genuine subsystem of the latter. This led the authors of the present paper to introduce *parsimonious linear logic*, a co-absorption-free subsystem of linear logic that nonetheless allows a stream-based interpretation of the exponentials.

We present two finitary proof systems for parsimonious linear logic: the system nuPLL, supporting non-uniform exponentials, and PLL, a fully uniform version. We investigate non-wellfounded counterparts of nuPLL and PLL, adapting to our setting the progressing criterion to maintain logical consistency. To recover the proof-theoretical behavior of nuPLL and PLL, we identify further global conditions on non-wellfounded proofs, that is, some forms of regularity to capture the notions of uniformity and non-uniformity. This leads us to two main non-wellfounded proof systems: *regular parsimonious linear logic* (rPLL^∞), defined via the regularity condition and corresponding to PLL, and *non-uniform parsimonious linear logic* (wrPLL^∞), defined via a *weak regularity* condition and corresponding to nuPLL.

The major contribution of this paper is the study of continuous cut-elimination in the setting of non-wellfounded parsimonious linear logic. We first introduce Scott-domains of partially defined non-wellfounded proofs, ordered by an approximation relation. Then, we define special infinitary proof rewriting strategies called *maximal and continuous infinitary cut-elimination strategies* (mc-ices) which compute (Scott-)continuous functions. Productivity in this framework is established by showing that, in presence of the progressing condition, these continuous functions return totally defined cut-free non-wellfounded proofs (Theorem 32.1). Moreover, we prove that they also preserve the (weak) progressing, the finite expandability, and the (weak) regularity conditions (Items 2–4 in Theorem 32).

[1] It is possible to construct linear logic proof systems with alternative (non equivalent) exponential modalities (see, e.g., [24]).

On a technical side, we stress that our methods and results distinguish from previous approaches to cut-elimination in a non-wellfounded setting in many respects. First, we get rid of many technical notions typically introduced to prove infinitary cut-elimination, such as the *multicut rule* or the *fairness conditions* (as in, e.g., [15, 3]), as these notions are subsumed by an *approximation-by-approximation* approach to cut-elimination. Furthermore, we prove productivity of cut-elimination and preservation of progressiveness in a more direct and constructive way, i.e., without going through auxiliary proof systems and avoiding arguments by contradiction (see, e.g., [3]). Finally, we prove for the first time preservation of regularity properties under continuous cut-elimination, essentially exploiting methods for compressing transfinite rewriting sequences to ω-long ones from [32, 29].

Finally, we define a denotational semantics for non-wellfounded parsimonious logic based on the relational model, with a standard multiset-based interpretation of the exponentials, and we show that this semantics is preserved under continuous cut-elimination (Theorem 37). We also prove that extending non-wellfounded parsimonious linear logic with digging prevents the existence of a cut-elimination result preserving the semantics (Theorem 39). Therefore, the impossibility of a stream-based definition of ! that validates digging (and contraction).

For lack of space, proofs are in the appendix if omitted or sketched in the body of the paper.

2 Preliminary notions

In this section we recall some basic notions from (non-wellfounded) proof theory, fixing the notation that will be adopted in this paper.

2.1 Derivations and coderivations

We assume that the reader is familiar with the syntax of sequent calculus, e.g. [33]. Here we specify some conventions adopted to simplify the content of this paper.

In this work we consider (**sequent**) **rules** of the form $\mathsf{r}\,\frac{}{\Gamma}$ or $\mathsf{r}\,\frac{\Gamma_1}{\Gamma}$ or $\mathsf{r}\,\frac{\Gamma_1 \quad \Gamma_2}{\Gamma}$, and we refer to the sequents Γ_1 and Γ_2 as the **premises**, and to the sequent Γ as the **conclusion** of the rule r. To avoid technicalities of the sequents-as-lists presentation, we follow [3] and we consider **sequents** as *sets of occurrences of formulas* from a given set of formulas. In particular, when we refer to a formula in a sequent we always consider a *specific occurrence* of it.

▶ **Definition 1.** *A (binary, possibly infinite)* ***tree*** $\mathcal{T}$ *is a subset of words in* $\{1,2\}^*$ *that contains the empty word* ϵ *(the* ***root*** *of* $\mathcal{T}$*) and is* ordered-prefix-closed *(i.e., if* $n \in \{1,2\}$ *and* $vn \in \mathcal{T}$*, then* $v \in \mathcal{T}$*, and if moreover* $v2 \in \mathcal{T}$*, then* $v1 \in \mathcal{T}$*). Elements of a tree* $\mathcal{T}$ *are called* ***nodes*** *and a node* $vn \in \mathcal{T}$ *with* $n \in \{1,2\}$ *is a* ***child*** *of* $v \in \mathcal{T}$*. Given a tree* $\mathcal{T}$ *and a node* $v \in \mathcal{T}$*, a* ***branch*** $\mathcal{B}$ *of* $\mathcal{T}$ *(from* v*) is a set of nodes in* $\mathcal{T}$ *of the form* vw *(for any* $w \in \{1,2\}^*$*) such that if they have at least one child in* $\mathcal{T}$ *then they have exactly one child in* $\mathcal{B}$*.*

A ***coderivation*** *over a set of rules* $\mathcal{S}$ *is a labeling* $\mathcal{D}$ *of a tree by sequents such that if* v *is a node with children* $v_1, \dots, v_n$ *(with* $n \in \{0,1,2\}$*), then there is an occurrence of a rule* r *in* $\mathcal{S}$ *with conclusion the sequent* $\mathcal{D}(v)$ *and premises the sequents* $\mathcal{D}(v_1), \dots, \mathcal{D}(v_n)$*. The* ***height*** *of* r *in* $\mathcal{D}$ *is the length of the node* $v \in \{1,2\}^*$ *such that* $\mathcal{D}(v)$ *is the conclusion of* r*.*

The ***conclusion*** *of* $\mathcal{D}$ *is the sequent* $\mathcal{D}(\epsilon)$*. If* v *is a node of the tree, the* ***sub-coderivation*** *of* $\mathcal{D}$ *rooted at* v *is the coderivation* $\mathcal{D}_v$ *defined by* $\mathcal{D}_v(w) = \mathcal{D}(vw)$*.*

A coderivation $\mathcal{D}$ *is* r***-free*** *(for a rule* $\mathsf{r} \in \mathcal{S}$*) if it contains no occurrence of* r*. It is* ***regular*** *if it has finitely many distinct sub-coderivations; it is* ***non-wellfounded*** *if it labels an infinite tree, and it is a* ***derivation*** *(with* ***size*** $|\mathcal{D}| \in \mathbb{N}$*) if it labels a finite tree (with* $|\mathcal{D}|$ *nodes).*

$$
{\scriptstyle\mathsf{ax}}\,\dfrac{}{A, A^{\perp}} \qquad {\scriptstyle\mathsf{cut}}\,\dfrac{\Gamma, A \quad A^{\perp}, \Delta}{\Gamma, \Delta} \qquad {\scriptstyle\parr}\,\dfrac{\Gamma, A, B}{\Gamma, A \parr B} \qquad {\scriptstyle\otimes}\,\dfrac{\Gamma, A \quad B, \Delta}{\Gamma, \Delta, A \otimes B} \qquad {\scriptstyle 1}\,\dfrac{}{1} \qquad {\scriptstyle\perp}\,\dfrac{\Gamma}{\Gamma, \perp} \qquad {\scriptstyle\mathsf{f!p}}\,\dfrac{\Gamma, A}{?\Gamma, !A} \qquad {\scriptstyle ?\mathsf{w}}\,\dfrac{\Gamma}{\Gamma, ?A} \qquad {\scriptstyle ?\mathsf{b}}\,\dfrac{\Gamma, A, ?A}{\Gamma, ?A}
$$

Figure 1 Sequent calculus rules of PLL.

$\underline{0}$	$\underline{1}$	$\mathcal{D}_{\mathsf{abs}}$	$\mathcal{D}_{\mathsf{der}}$
${\scriptstyle\parr}\dfrac{{\scriptstyle\parr}\dfrac{{\scriptstyle ?\mathsf{w}}\dfrac{{\scriptstyle\mathsf{ax}}\dfrac{}{X^{\perp}, X}}{?(X \otimes X^{\perp}), X^{\perp}, X}}{?(X \otimes X^{\perp}), X^{\perp} \parr X}}{?(X \otimes X^{\perp}) \parr X^{\perp} \parr X}$	${\scriptstyle\parr\times 2}\dfrac{{\scriptstyle ?\mathsf{b}}\dfrac{{\scriptstyle ?\mathsf{w}}\dfrac{{\scriptstyle\otimes}\dfrac{{\scriptstyle\mathsf{ax}}\dfrac{}{X^{\perp}, X} \quad {\scriptstyle\mathsf{ax}}\dfrac{}{X^{\perp}, X}}{X \otimes X^{\perp}, X^{\perp}, X}}{?(X \otimes X^{\perp}), X \otimes X^{\perp}, X^{\perp}, X}}{?(X \otimes X^{\perp}), X^{\perp}, X}}{?(X \otimes X^{\perp}) \parr X^{\perp} \parr X}$	${\scriptstyle\parr}\dfrac{{\scriptstyle ?\mathsf{b}}\dfrac{{\scriptstyle\otimes}\dfrac{{\scriptstyle\mathsf{ax}}\dfrac{}{A^{\perp}, A} \quad {\scriptstyle\mathsf{ax}}\dfrac{}{?A^{\perp}, !A}}{A^{\perp}, ?A^{\perp}, A \otimes !A}}{?A^{\perp}, A \otimes !A}}{?A^{\perp} \parr (A \otimes !A)}$	${\scriptstyle\parr}\dfrac{{\scriptstyle ?\mathsf{b}}\dfrac{{\scriptstyle ?\mathsf{w}}\dfrac{{\scriptstyle\mathsf{ax}}\dfrac{}{A^{\perp}, A}}{A^{\perp}, ?A^{\perp}, A}}{?A^{\perp}, A}}{?A^{\perp} \parr A}$

Figure 2 Examples of derivations in PLL.

Given a set of coderivations X*, a sequent* Γ *is* ***provable*** *in* X *(noted* $\vdash_{\mathsf{X}} \Gamma$*) if there is a coderivation in* X *with conclusion* Γ*.*

While derivations are usually represented as finite trees, regular coderivations can be represented as *finite* directed (possibly cyclic) graphs: a cycle is created by linking the roots of two identical subcoderivations.

▶ **Definition 2** (Bar). *Let* $\mathcal{D}$ *be a coderivation. A set* $\mathcal{V}$ *of nodes in* $\mathcal{D}$ *is a* ***bar*** *(of* $\mathcal{D}$*) if:*

- *any infinite branch of* $\mathcal{D}$ *contains a node in* $\mathcal{V}$*;*
- *any pair of nodes in* $\mathcal{V}$ *are mutually incomparable (w.r.t. the partial order in* $\mathcal{D}$*).*

We say that a bar $\mathcal{V}$ *has* ***height*** h *if every node in* $\mathcal{V}$ *that is not a leaf of* $\mathcal{D}$ *has height* $\geq h$*.*

3 Parsimonious Linear Logic

In this paper we consider the set of **formulas** for propositional multiplicative-exponential linear logic with units (MELL). These are generated by a countable set of propositional variables $\mathcal{A} = \{X, Y, \ldots\}$ using the following grammar:

$$A, B ::= X \mid X^{\perp} \mid A \otimes B \mid A \parr B \mid !A \mid ?A \mid 1 \mid \perp$$

A **!-formula** (resp. **?-formula**) is a formula of the form $!A$ (resp. $?A$). **Linear negation** $(\cdot)^{\perp}$ is defined by De Morgan's laws $(A^{\perp})^{\perp} = A$, $(A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp}$, $(!A)^{\perp} = ?A^{\perp}$, and $(1)^{\perp} = \perp$ while **linear implication** is defined as $A \multimap B := A^{\perp} \parr B$.

▶ **Definition 3.** ***Parsimonious linear logic****, denoted by* PLL*, is the set of rules in Figure 1, that is,* ***axiom*** *(*ax*),* ***cut*** *(*cut*),* ***tensor*** *(*$\otimes$*),* ***par*** *(*$\parr$*),* ***one*** *(*1*),* ***bottom*** *(*$\perp$*),* ***functorial promotion*** *(*f!p*),* ***weakening*** *(*?w*),* ***absorption*** *(*?b*). Rules* ax*,* $\otimes$*,* $\parr$*,* 1 *and* $\perp$ *are called* ***multiplicative****, while rules* f!p*,* ?w *and* ?b *are called* ***exponential****. We also denote by* PLL *the set of derivations over the rules in* PLL*.*

▶ **Example 4.** Figure 2 gives some examples of derivation in PLL. The (distinct) derivations $\underline{0}$ and $\underline{1}$ prove the same formula $\mathbf{N} := !(X \multimap X) \multimap X \multimap X$. The derivation $\mathcal{D}_{\mathsf{abs}}$ proves the *absorption law* $!A \multimap A \otimes !A$; the derivation $\mathcal{D}_{\mathsf{der}}$ proves the *dereliction law* $!A \multimap A$.

The **cut-elimination** relation $\rightarrow_{\mathsf{cut}}$ in PLL is the union of **principal** cut-elimination steps in Figure 3 (**multiplicative**) and Figure 4 (**exponential**) and **commutative** cut-elimination steps in Figure 5. The reflexive-transitive closure of $\rightarrow_{\mathsf{cut}}$ is noted $\rightarrow^{*}_{\mathsf{cut}}$.

$$\dfrac{\dfrac{}{A, A^{\perp}}\,\mathsf{ax} \qquad \Gamma, A}{\Gamma, A}\,\mathsf{cut} \to_{\mathsf{cut}} \Gamma, A \qquad\qquad \dfrac{\dfrac{\Gamma, A, B}{\Gamma, A ⅋ B}\,⅋ \qquad \dfrac{\Delta, A^{\perp} \quad B^{\perp}, \Sigma}{\Delta, A^{\perp} \otimes B^{\perp}, \Sigma}\,\otimes}{\Gamma, \Delta, \Sigma}\,\mathsf{cut} \to_{\mathsf{cut}} \dfrac{\dfrac{\Gamma, B, A \quad A^{\perp}, \Delta}{\Gamma, \Delta, B}\,\mathsf{cut} \qquad B^{\perp}, \Sigma}{\Gamma, \Delta, \Sigma}\,\mathsf{cut} \qquad\qquad \dfrac{\dfrac{\Gamma}{\Gamma, \bot}\,\bot \qquad \dfrac{}{1}\,1}{\Gamma}\,\mathsf{cut} \to_{\mathsf{cut}} \Gamma$$

Figure 3 Multiplicative cut-elimination steps in PLL.

$$\dfrac{\dfrac{\Gamma, A}{?\Gamma, !A}\,\mathsf{f!p} \qquad \dfrac{A^{\perp}, \Delta, B}{?A^{\perp}, ?\Delta, !B}\,\mathsf{f!p}}{?\Gamma, ?\Delta, !B}\,\mathsf{cut} \to_{\mathsf{cut}} \dfrac{\dfrac{\Gamma, A \quad A^{\perp}, \Delta, B}{\Gamma, \Delta, B}\,\mathsf{cut}}{?\Gamma, ?\Delta, !B}\,\mathsf{f!p} \qquad\qquad \dfrac{\dfrac{\Gamma, A}{?\Gamma, !A}\,\mathsf{f!p} \qquad \dfrac{\Delta}{\Delta, ?A^{\perp}}\,?\mathsf{w}}{?\Gamma, \Delta}\,\mathsf{cut} \to_{\mathsf{cut}} \dfrac{\Delta}{?\Gamma, \Delta}\,?\mathsf{w}$$

$$\dfrac{\dfrac{\Gamma, A}{?\Gamma, !A}\,\mathsf{f!p} \qquad \dfrac{\Delta, A^{\perp}, ?A^{\perp}}{\Delta, ?A^{\perp}}\,?\mathsf{b}}{?\Gamma, \Delta}\,\mathsf{cut} \to_{\mathsf{cut}} \dfrac{\dfrac{\Gamma, A \qquad \dfrac{\dfrac{\Gamma, A}{?\Gamma, !A}\,\mathsf{f!p} \quad \Delta, A^{\perp}, ?A^{\perp}}{?\Gamma, \Delta, A^{\perp}}\,\mathsf{cut}}{\Gamma, ?\Gamma, \Delta}\,\mathsf{cut}}{?\Gamma, \Delta}\,?\mathsf{b}$$

Figure 4 Exponential cut-elimination steps in PLL.

▶ **Theorem 5.** *For every* $\mathcal{D} \in \mathsf{PLL}$*, there is a cut-free* $\mathcal{D}' \in \mathsf{PLL}$ *such that* $\mathcal{D} \to^{*}_{\mathsf{cut}} \mathcal{D}'$*.*

Sketch of proof. We associate with any derivation $\mathcal{D}$ in PLL a derivation $\mathcal{D}^{♠}$ in MELL sequent calculus. Thanks to additional commutative cut-elimination steps, we prove that cut-elimination in MELL rewrites $\mathcal{D}^{♠}$ to the translation of a derivation in PLL. The termination of cut-elimination in PLL follows from the result in MELL [26]. Details are in Appendix A. ◀

Akin to light linear logic [18, 20, 28], the exponential rules of PLL are weaker than those in MELL: the usual promotion rule is replaced by f!p (*functorial promotion*), and the usual contraction and dereliction rules by ?b. As a consequence, the *digging* formula $!A \multimap !!A$ and the *contraction* formula $!A \multimap !A \otimes !A$ are not provable in PLL (unlike the dereliction formula, Example 4). This allows us to interpret computationally these weaker exponentials in terms of streams, as well as to control the complexity of cut-elimination [22, 23].

It is easy to show that MELL = PLL + digging: if we add the digging formula as an axiom (or equivalently, the *digging rule* ??d in Figure 12) to the set of rules in Figure 1, then the contraction formula becomes provable, and the obtained proof system coincides with MELL.

4 Non-wellfounded Parsimonious Linear Logic

In linear logic, a formula $!A$ is interpreted as the availability of A at will. This intuition still holds in PLL. Indeed, the Curry-Howard correspondence interprets rule f!p introducing the modality ! as an operator taking a derivation $\mathcal{D}$ of A and creating a (infinite) *stream* $(\mathcal{D}, \mathcal{D}, \dots, \mathcal{D}, \dots)$ of copies of the proof $\mathcal{D}$. Each element of the stream is accessed via the cut-elimination step f!p vs ?b in Figure 4: rule ?b is interpreted as an operator *popping* one copy of $\mathcal{D}$ out of the stream. Pushing these ideas further, Mazza [22] introduced *parsimonious logic* **PL**, a type system (comprising rules f!p and ?b) characterizing the logspace decidable problems.

Mazza and Terui then introduced in [23] another type system, $\mathbf{nuPL}_{\forall\ell}$, based on parsimonious logic and capturing the complexity class $\mathbf{P}/\mathsf{poly}$ (i.e., the problems decidable by polynomial size families of boolean circuits [2]). Their system is endowed with a *non-uniform* version of the functorial promotion, which takes a finite set of proofs $\mathcal{D}_1, \dots, \mathcal{D}_n$ of A and a (possibly non-recursive) function $f \colon \mathbb{N} \to \{1, \dots, n\}$ as premises, and constructs a proof of $!A$ modeling the stream $(\mathcal{D}_{f(0)}, \mathcal{D}_{f(1)}, \dots, \mathcal{D}_{f(n)}, \dots)$. This typing rule is the key tool to encode the so-called *advices* for Turing machines, an essential step to show completeness for $\mathbf{P}/\mathsf{poly}$.

$$\dfrac{\dfrac{\Gamma_1, A}{\Gamma, A}\,\mathsf{r} \quad A^\perp, \Delta}{\Gamma, \Delta}\,\mathsf{cut} \;\to_{\mathsf{cut}}\; \dfrac{\dfrac{\Gamma_1, A \quad A^\perp, \Delta}{\Gamma_1, \Delta}\,\mathsf{cut}}{\Gamma, \Delta}\,\mathsf{r} \qquad \dfrac{\dfrac{\Gamma_1, A \quad \Gamma_2}{\Gamma, A}\,\mathsf{r} \quad \Delta, A^\perp}{\Gamma, \Delta}\,\mathsf{cut} \;\to_{\mathsf{cut}}\; \dfrac{\dfrac{\Gamma_1, A \quad A^\perp, \Delta}{\Gamma_1, \Delta}\,\mathsf{cut} \quad \Gamma_2}{\Gamma, \Delta}\,\mathsf{r}$$

Figure 5 Commutative cut-elimination steps in PLL, where $\mathsf{r} \neq \mathsf{cut}$.

In a similar vein, we can endow PLL with a non-uniform version of f!p called **infinitely branching promotion** (ib!p), which constructs a stream $(\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_n, \ldots)$ with finite support, i.e., made of *finitely* many distinct derivations (of the same conclusion):[2]

$$\mathsf{ib!p}\,\dfrac{\overset{\mathcal{D}_0}{\Gamma, A} \quad \overset{\mathcal{D}_1}{\Gamma, A} \quad \cdots \quad \overset{\mathcal{D}_n}{\Gamma, A} \quad \cdots}{?\Gamma, !A}\;\{\mathcal{D}_i \mid i \in \mathbb{N}\}\text{ is finite} \quad \Bigg| \quad \mathsf{!w}\,\dfrac{}{!A} \qquad \mathsf{!b}\,\dfrac{\Gamma, A \quad \Delta, !A}{\Gamma, \Delta, !A} \tag{1}$$

The side condition on ib!p provides a proof theoretic counterpart to the function $f \colon \mathbb{N} \to \{1, \ldots, n\}$ in $\mathbf{nuPL}_{\forall\ell}$. Clearly, f!p is subsumed by the rule ib!p, as it corresponds to the special (uniform) case where $\mathcal{D}_i = \mathcal{D}_{i+1}$ for all $i \in \mathbb{N}$.

▶ **Definition 6.** *We define the set of rules* nuPLL $:= \{\mathsf{ax}, \otimes, ⅋, 1, \bot, \mathsf{cut}, ?\mathsf{b}, ?\mathsf{w}, \mathsf{ib!p}\}$. *We also denote by* nuPLL *the set of derivations over the rules in* nuPLL.[3]

There are some notable differences between nuPLL and Mazza and Terui's original system $\mathbf{nuPL}_{\forall\ell}$ [23]. As opposed to nuPLL, $\mathbf{nuPL}_{\forall\ell}$ is formulated as an intuitionistic (type) system. Furthermore, to achieve completeness for **P**/poly, these authors introduced second-order quantifiers and the co-absorption (!b) and co-weakening (!w) rules displayed in (1).

Cut-elimination steps for nuPLL are in Figures 3, 5, and 16 (Figure 16 is in Appendix A because we do not use it: it just adapts the exponential steps to ib!p). In particular, the step ib!p-vs-?b in Figure 16 *pops* the first premise $\mathcal{D}_0$ of ib!p out of the stream $(\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_n, \ldots)$.

4.1 From infinitely branching proofs to non-wellfounded proofs

In this paper we explore a dual approach to the one of $\mathbf{nuPL}_{\forall\ell}$ (and nuPLL): instead of considering (wellfounded) derivations with infinite branching, we consider (non-wellfounded) coderivations with finite branching. For this purpose, the infinitary rule ib!p of nuPLL is replaced by the binary rule below, called **conditional promotion** (c!p):

$$\mathsf{c!p}\,\dfrac{\Gamma, A \quad ?\Gamma, !A}{?\Gamma, !A} \tag{2}$$

▶ **Definition 7.** *We define the set of rules* $\mathsf{PLL}^\infty := \{\mathsf{ax}, \otimes, ⅋, 1, \bot, \mathsf{cut}, ?\mathsf{b}, ?\mathsf{w}, \mathsf{c!p}\}$. *We also denote by* PLL^∞ *the set of coderivations over the rules in* PLL^∞.

In other words, PLL^∞ is the set of coderivations generated by the same rules as PLL, except that f!p is replaced by c!p. From now on, we will only consider coderivations in PLL^∞.

[2] Rule ib!p is reminiscent of the ω-rule used in (first-order) Peano arithmetic to derive formulas of the form $\forall x \phi$ that cannot be proven in a uniform way.

[3] To be rigorous, this requires a slight change in Definition 1: the tree labeled by a derivation in nuPLL must be over $\mathbb{N}^\omega$ instead of $\{1, 2\}^*$, in order to deal with infinitely branching derivations.

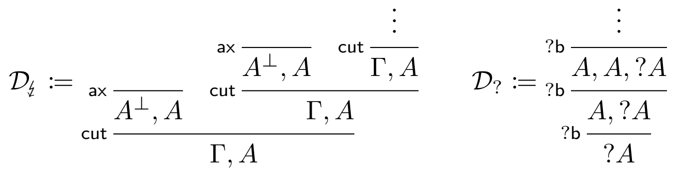


Figure 6 Two non-wellfounded and non-progressing coderivations in PLL^∞.

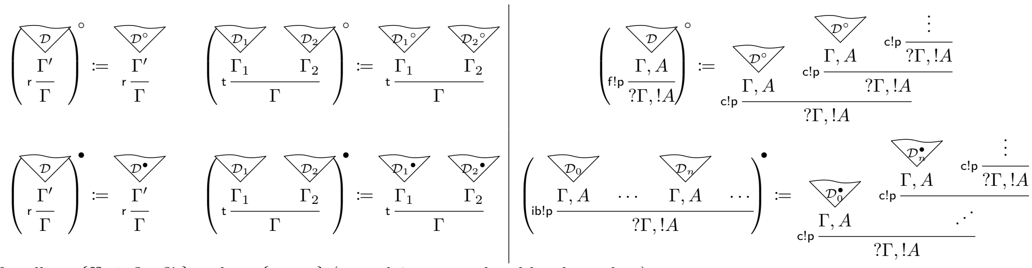


Figure 7 Translations $(\cdot)^\circ$ from PLL to PLL^∞, and $(\cdot)^\bullet$ from nuPLL to PLL^∞.

▶ **Example 8.** Figure 6 shows two non-wellfounded coderivations in PLL^∞: $\mathcal{D}_{\not\,}$ (resp. $\mathcal{D}_?$) has an infinite branch of cut (resp. $?\mathsf{b}$) rules, and is (resp. is not) regular.

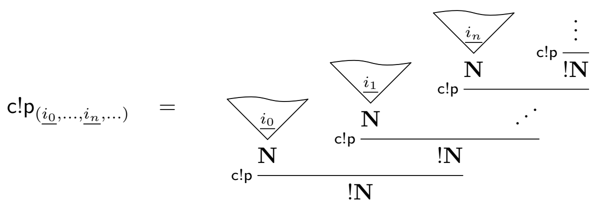
We can embed PLL and nuPLL into PLL^∞ via the conclusion-preserving translations $(\cdot)^\circ\colon \mathsf{PLL} \to \mathsf{PLL}^\infty$ and $(\cdot)^\bullet\colon \mathsf{nuPLL} \to \mathsf{PLL}^\infty$ defined in Figure 7 by induction on derivations: they map all rules to themselves except $\mathsf{f!p}$ and $\mathsf{ib!p}$, which are "unpacked" into non-wellfounded coderivations that iterate infinitely many times the rule $\mathsf{c!p}$.

An infinite chain of $\mathsf{c!p}$ rules (Figure 8) is a structure of interest in itself in PLL^∞.

▶ **Definition 9.** *A* ***non-wellfounded box*** *(*nwb *for short) is a coderivation $\mathcal{D}$ with an infinite branch $\{\epsilon, 2, 22, \dots\}$ (the* ***main branch*** *of $\mathcal{D}$) all labeled by* $\mathsf{c!p}$ *rules as in Figure 8, where $!A$ in the conclusion is the* ***principal formula*** *of $\mathcal{D}$, and $\mathcal{D}_0, \mathcal{D}_1, \dots$ are the* ***calls*** *of $\mathcal{D}$. We denote $\mathcal{D}$ by $\mathsf{c!p}_{(\mathcal{D}_0,\dots,\mathcal{D}_n,\dots)}$.*

Let $\mathfrak{S} = \mathsf{c!p}_{(\mathcal{D}_0,\dots,\mathcal{D}_n,\dots)}$ be a nwb*. We may write $\mathfrak{S}(i)$ to denote $\mathcal{D}_i$. We say that $\mathfrak{S}$ has* ***finite support*** *(resp. is* ***periodic*** *with* ***period*** *k) if $\{\mathfrak{S}(i) \mid i \in \mathbb{N}\}$ is finite (resp. if $\mathfrak{S}(i) = \mathfrak{S}(k+i)$ for any $i \in \mathbb{N}$). A coderivation $\mathcal{D}$ has* ***finite support*** *(resp. is* ***periodic****) if any* nwb *in $\mathcal{D}$ has finite support (resp. is periodic).*

▶ **Example 10.** Consider the following nwb of the formula $!\mathbf{N}$, where $\mathbf{N} := !(X \multimap X) \multimap X \multimap X$ has at two distinct derivations $\underline{\mathbf{0}}$ and $\underline{\mathbf{1}}$ (Example 4), and $i_j \in \{\mathbf{0}, \mathbf{1}\}$ for all $j \in \mathbb{N}$.



(3)

Thus $\mathsf{c!p}_{(\underline{i_0},\dots,\underline{i_n},\dots)}$ has finite support, as its only calls can be $\underline{\mathbf{0}}$ or $\underline{\mathbf{1}}$, and it is periodic if and only if so is the infinite sequence $(i_0, \dots, i_n, \dots) \in \{\mathbf{0}, \mathbf{1}\}^\omega$.

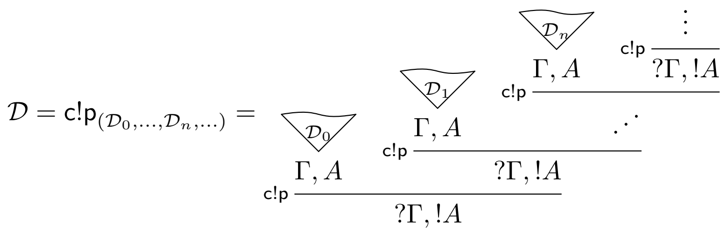


Figure 8 A non-wellfounded box in PLL^∞.

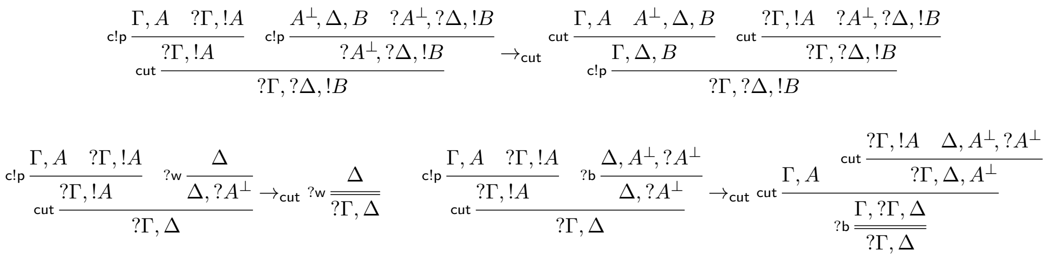


Figure 9 Exponential cut-elimination steps for coderivations of PLL^∞.

The *cut-elimination* steps $\to_{\mathsf{cut}}$ for PLL^∞ are in Figures 3, 5, and 9. Computationally, they allow the $\mathsf{c!p}$ rule to be interpreted as a *coinductive* definition of a stream of type $!A$ from a stream of the same type to which an element of type A is prepended. In particular, the cut-elimination step $\mathsf{c!p}$ vs $?\mathsf{b}$ accesses the head of a stream: rule $?\mathsf{b}$ acts as a *popping* operator.

As a consequence, the nwb in Figure 8 constructs a stream $(\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_n, \dots)$ similarly to $\mathsf{ib!p}$ but, unlike the latter, all the $\mathcal{D}_i$'s may be pairwise distinct. The reader expert in linear logic can see a nwb as a box with possibly *infinitely many* distinct contents (its calls), while usual linear logic boxes (and $\mathsf{f!p}$ in PLL) provide infinitely many copies of the *same* content.

Rules $\mathsf{f!p}$ in PLL and $\mathsf{ib!p}$ in nuPLL are mapped by $(\cdot)^\circ$ and $(\cdot)^\bullet$ into nwbs, which are non-wellfounded coderivations. Hence, the cut-elimination steps $\mathsf{f!p}$ vs $\mathsf{f!p}$ in PLL and $\mathsf{ib!p}$ vs $\mathsf{ib!p}$ in nuPLL can only be simulated by infinitely many cut-elimination steps in PLL^∞.

Note that $\mathcal{D}_{\lightning} \in \mathsf{PLL}^\infty$ in Figure 6 is not cut-free, and if $\mathcal{D}_{\lightning} \to_{\mathsf{cut}} \mathcal{D}$ then $\mathcal{D} = \mathcal{D}_{\lightning}$: thus $\mathcal{D}_{\lightning}$ cannot reduce to a cut-free coderivation, and so the cut-elimination theorem fails in PLL^∞.

4.2 Consistency via a progressing criterion

In a non-wellfounded setting such as PLL^∞, any sequent is provable. Indeed, the (non-wellfounded) coderivation $\mathcal{D}_{\lightning}$ in Figure 6 shows that any non-empty sequent (in particular, any formula) is provable in PLL^∞, and the empty sequent is provable in PLL^∞ by applying the cut rule on the conclusions B and $B^\perp$ (for any formula B) of two derivations $\mathcal{D}_{\lightning}$.

The standard way to recover logical consistency in non-wellfounded proof theory is to introduce a global soundness condition on coderivations, called **progressing criterion**. In PLL^∞, this criterion relies on tracking occurrences of $!$-formulas in a coderivation.

▶ **Definition 11.** *Let $\mathcal{D}$ be a coderivation in PLL^∞. It is **weakly progressing** if every infinite branch contains infinitely many right premises of $\mathsf{c!p}$-rules.*

*An occurrence of formula in a premise of a rule r is the **parent** of an occurrence of a formula in the conclusion if they are connected according to the edges depicted in Figure 10.*

*A $!$-**thread** (resp. $?$-**thread**) in $\mathcal{D}$ is a maximal sequence $(A_i)_{i \in I}$ of $!$-formulas (resp. $?$-formulas) for some downward-closed $I \subseteq \mathbb{N}$ such that A_{i+1} is the parent of A_i for all $i \in I$. A $!$-thread $(A_i)_{i \in I}$ is **progressing** if A_j is in the conclusion of a $\mathsf{c!p}$ for infinitely many $j \in I$.*

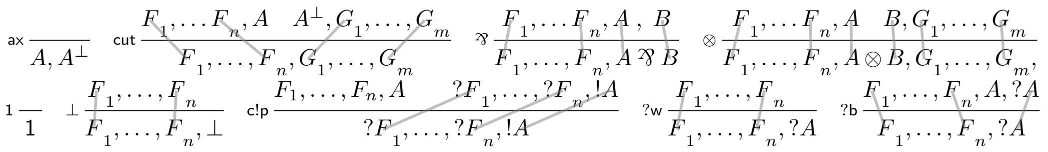
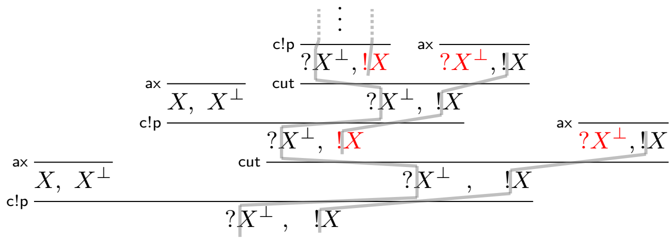


Figure 10 PLL^∞ rules: edges connect a formula in the conclusion with its parent(s) in a premise.

*$\mathcal{D}$ is **progressing** if every infinite branch contains a progressing !-thread. We define* pPLL^∞ *(resp.* wpPLL^∞*) as the set of progressing (resp. weak-progressing) coderivations in* PLL^∞*.*

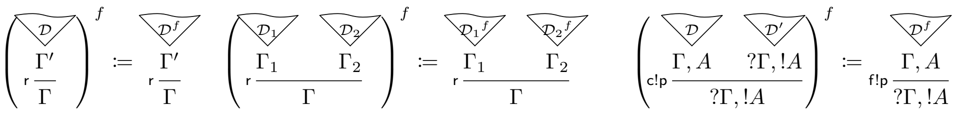
▶ Remark 12. Clearly, any progressing coderivation is weakly progressing too, but the converse fails (Example 13), therefore $\mathsf{pPLL}^\infty \subsetneq \mathsf{wpPLL}^\infty$. Moreover, the main branch of any nwb contains by definition a progressing !-thread of its principal formula.

▶ **Example 13.** Coderivations in Figure 6 are not weakly progressing (hence, not progressing): the rightmost branch of $\mathcal{D}_{\not{\iota}}$, i.e., the branch $\{\epsilon, 2, 22, \dots\}$, and the unique branch of $\mathcal{D}_?$ are infinite and contain no $\mathsf{c!p}$-rules. In contrast, the nwb $\mathsf{c!p}_{(\underline{i_0},\dots,\underline{i_n},\dots)}$ in Example 10 is progressing by Remark 12, since its main branch is the only infinite branch. Below, a regular, weakly progressing but not progressing coderivation ($!X$ in the conclusion of $\mathsf{c!p}$ is a cut formula, so the branch $\{\epsilon, 2, 21, 212, 2121, \dots\}$ is infinite but has no progressing !-thread).



▶ **Lemma 14.** *Let Γ be a sequent. Then, $\vdash_{\mathsf{PLL}} \Gamma$ if and only if $\vdash_{\mathsf{wpPLL}^\infty} \Gamma$.*

Proof. Given $\mathcal{D} \in \mathsf{PLL}$, $\mathcal{D}^\bullet \in \mathsf{PLL}^\infty$ preserves the conclusion and is progressing, hence weakly progressing (see Remark 12). Conversely, given a weakly progressing coderivation $\mathcal{D}$, we define a derivation $\mathcal{D}^f \in \mathsf{PLL}$ with the same conclusion by applying, bottom-up, the translation:



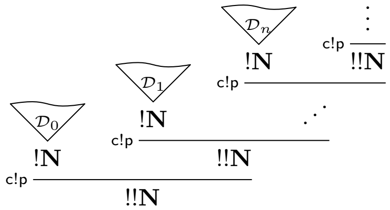
with $\mathsf{r} \neq \mathsf{c!p}$. Note that the derivation $\mathcal{D}^f$ is well-defined because $\mathcal{D}$ is weakly progressing. ◀

▶ **Corollary 15.** *The empty sequent is not provable in* wpPLL^∞ *(and hence in* pPLL^∞*).*

Proof. If the empty sequent were provable in wpPLL^∞, then there would be a cut-free derivation $\mathcal{D} \in \mathsf{PLL}$ of the empty sequent by Lemma 14 and Theorem 5, but this is impossible since cut is the only rule in PLL that could have the empty sequent in its conclusion. ◀

4.3 Recovering (weak forms of) regularity

The progressing criterion cannot capture the finiteness condition of the rule $\mathsf{ib!p}$ in the derivations in nuPLL. By means of example, consider the nwb below, which is progressive but cannot be the image of the rule $\mathsf{ib!p}$ via $(\cdot)^\bullet$ (see Figure 7) since $\{\mathcal{D}_i \mid i \in \mathbb{N}\}$ is infinite.



with $\mathcal{D}_i = \mathsf{c!p}_{(\underbrace{\underline{\mathbf{1}},\dots,\underline{\mathbf{1}}}_{i},\underline{\mathbf{0}},\dots)}$ for each $i \in \mathbb{N}$. (4)

To identify in pPLL^∞ the coderivations corresponding to derivations in nuPLL and in PLL via the translations $(\cdot)^\bullet$ and $(\cdot)^\circ$, respectively, we need additional conditions.

▶ **Definition 16.** *A coderivation is* ***weakly regular*** *if it has only finitely many distinct sub-coderivations whose conclusions are left premises of* $\mathsf{c!p}$*-rules; it is* ***finitely expandable*** *if any branch contains finitely many* cut *and* $?\mathsf{b}$ *rules. We denote by* wrPLL^∞ *(resp.* rPLL^∞*) the set of weakly regular (resp. regular) and finitely expandable coderivations in* pPLL^∞*.*

▶ Remark 17. Regularity implies weak regularity and the converse fails as shown in Example 18 below, therefore $\mathsf{rPLL}^\infty \subsetneq \mathsf{wrPLL}^\infty$. Moreover, $\mathcal{D} \in \mathsf{PLL}^\infty$ is regular (resp. weakly regular) if and only if any nwb in $\mathcal{D}$ is periodic (resp. has finite support).

▶ **Example 18.** Coderivations $\mathcal{D}_{\not{\,}}$ and $\mathcal{D}_?$ in Figure 6 are not finitely expandable, as their infinite branch has infinitely many cut or $?\mathsf{b}$, but they are weakly regular, since they have no $\mathsf{c!p}$ rules. The coderivation in (4) is not weakly regular because $\{\mathcal{D}_i \mid i \in \mathbb{N}\}$ is infinite.

An example of a weakly regular but not regular coderivation is the nwb $\mathsf{c!p}_{(\underline{i_0},\dots,\underline{i_n},\dots)}$ in Example 10 when the infinite sequence $(i_j)_{j\in\mathbb{N}} \in \{\mathbf{0},\mathbf{1}\}^\omega$ is not periodic: in each rule $\mathsf{c!p}$ there, the left premise can only be $\underline{\mathbf{0}}$ or $\underline{\mathbf{1}}$ (so the nwb is weakly regular), but the right premise is a distinct coderivation (so the nwb is not regular). Moreover, that nwb is finitely expandable since it contains no $?\mathsf{b}$ or cut.

The sets rPLL^∞ and wrPLL^∞ are the non-wellfounded counterparts of PLL and nuPLL, respectively. Indeed, we have the following correspondence via the translations $(\cdot)^\circ$ and $(\cdot)^\bullet$.

▶ **Proposition 19.**
1. *If* $\mathcal{D} \in \mathsf{PLL}$ *(resp.* $\mathcal{D} \in \mathsf{nuPLL}$*) with conclusion* Γ*, then* $\mathcal{D}^\circ \in \mathsf{rPLL}^\infty$ *(resp.* $\mathcal{D}^\bullet \in \mathsf{wrPLL}^\infty$*) with conclusion* Γ*, and every* $\mathsf{c!p}$ *in* $\mathcal{D}^\circ$ *(resp.* $\mathcal{D}^\bullet$*) belongs to a* nwb*.*
2. *If* $\mathcal{D}' \in \mathsf{rPLL}^\infty$ *(resp.* $\mathcal{D}' \in \mathsf{wrPLL}^\infty$*) and every* $\mathsf{c!p}$ *in* $\mathcal{D}'$ *belongs to a* nwb*, then there is* $\mathcal{D} \in \mathsf{PLL}$ *(resp.* $\mathcal{D} \in \mathsf{nuPLL}$*) such that* $\mathcal{D}^\circ = \mathcal{D}'$ *(resp.* $\mathcal{D}^\bullet = \mathcal{D}'$*).*

Progressing and weak progressing coincide in finite expandable coderivations.

▶ **Lemma 20.** *Let* $\mathcal{D} \in \mathsf{PLL}^\infty$ *be finitely expandable. If* $\mathcal{D} \in \mathsf{wpPLL}^\infty$ *then any infinite branch contains the principal branch of a* nwb*. Moreover,* $\mathcal{D} \in \mathsf{pPLL}^\infty$ *iff* $\mathcal{D} \in \mathsf{wpPLL}^\infty$*.*

Proof. Let $\mathcal{D} \in \mathsf{wpPLL}^\infty$ be finitely expandable, and let $\mathcal{B}$ be an infinite branch in $\mathcal{D}$. By finite expandability there is $h \in \mathbb{N}$ such that $\mathcal{B}$ contains no conclusion of a cut or $?\mathsf{b}$ with height greater than h. Moreover, by weakly progressing there is an infinite sequence $h \leq h_0 < h_1 < \ldots < h_n < \ldots$ such that the sequent of $\mathcal{B}$ at height h_i has shape $?\Gamma_i, !A_i$. By inspecting the rules in Figure 1, each such $?\Gamma_i, !A_i$ can be either the conclusion of either a $?\mathsf{w}$ or a $\mathsf{c!p}$ (with right premise $?\Gamma_i, !A_i$). So, there is a k large enough such that, for any $i \geq k$, only the latter case applies (and, in particular, $\Gamma_i = \Gamma$ and $A_i = A$ for some Γ, A). Therefore, h_k is the root of a nwb. This also shows $\mathcal{D} \in \mathsf{pPLL}^\infty$. By Remark 12, $\mathsf{pPLL}^\infty \subseteq \mathsf{wpPLL}^\infty$. ◀

By inspecting the steps in Figures 3, 5, and 9, we prove the following preservations.

▶ **Proposition 21.** *Cut elimination preserves weak-regularity, regularity and finite expandability. Therefore, if* $\mathcal{D} \in \mathsf{X}$ *with* $\mathsf{X} \in \{\mathsf{rPLL}^\infty, \mathsf{wrPLL}^\infty\}$ *and* $\mathcal{D} \to_{\mathsf{cut}} \mathcal{D}'$*, then also* $\mathcal{D}' \in \mathsf{X}$*.*

5 Continuous cut-elimination

Cut-elimination for (finitary) sequent calculi proceeds by introducing a proof rewriting strategy that stepwise decreases an appropriate termination ordering (see, e.g, [33]). Typically, these proof rewriting strategies consist on pushing upward the topmost cuts via the cut-elimination steps in order to eventually eliminate them.

A somewhat dual approach is investigated in the context of non-wellfounded proofs [3, 16]. It consists on *infinitary* proof rewriting strategies that gradually push upward the bottommost cuts. In this setting, the progressing condition is essential to guarantee *productivity*, i.e., that such proof rewriting strategies construct strictly increasing approximations of the cut-free proof, which can thus be obtained as a (well-defined) *limit*.

A major obstacle of this approach arises when the bottommost cut r is below another one r'. In this case, no cut-elimination step can be applied to r, so proof rewriting runs into an apparent stumbling block. To circumvent this problem, in [3, 16] a special cut-elimination step is introduced, which merges r and r' in a single, generalized cut rule called *multicut*.

In this section we study a continuous cut-elimination method that does not rely on multicut rules, following an alternative idea in which the notion of approximation plays an even more central rule, inspired by the topological approaches to infinite trees [6]. To this end, we assume the reader familiar with basic definitions on domain-theory (see, e.g., [1]).

5.1 Approximating coderivations

In this subsection we introduce *open coderivations*, which approximate coderivations. Open coderivations form Scott-domains, on top of which we will formally define continuous cut elimination. Furthermore, we exploit open coderivations to present a decomposition result for finitely expandable and progressing coderivations.

▶ **Definition 22.** *We define the set of rules* $\mathsf{oPLL}^\infty := \mathsf{PLL}^\infty \cup \{\mathsf{hyp}\}$*, where* $\mathsf{hyp} := \mathsf{hyp}\,\frac{}{\Gamma}$ *for any sequent* Γ.[4] *We will also refer to* oPLL^∞ *as the set of coderivations over* oPLL^∞*, which we call* ***open coderivations****. An open coderivation is* ***normal*** *if no cut-elimination step can be applied to it, that is, if one premise of each* cut *is a* hyp*. An* ***open derivation*** *is a derivation in* oPLL^∞*. We denote by* $\mathsf{oPLL}^\infty(\Gamma)$ *the set of open coderivations with conclusion* Γ*, and by* $\mathcal{K}(\mathcal{D})$ *the set of finite approximations of* $\mathcal{D}$*.*

▶ **Definition 23.** *Let* $\mathcal{D}$ *be an open coderivation,* $\mathcal{V} \subseteq \{1,2\}^*$ *be a set of mutually incomparable (w.r.t. the prefix order) nodes of* $\mathcal{D}$*, and* $\{\mathcal{D}'_\nu\}_{\nu\in\mathcal{V}}$ *be a set of open coderivations where* $\mathcal{D}'_\nu$ *has the same conclusion as the subderivation* $\mathcal{D}_\nu$ *of* $\mathcal{D}$*. We denote by* $\mathcal{D}\{\mathcal{D}'_\nu/\nu\}_{\nu\in\mathcal{V}} = \mathcal{D}(\mathcal{D}'_{\nu_1}/\nu_1, \dots, \mathcal{D}'_{\nu_n}/\nu_n)$*, the open coderivation obtained by replacing each* $\mathcal{D}_\nu$ *with* $\mathcal{D}'_\nu$*.*

The ***pruning*** *of* $\mathcal{D}$ *over* $\mathcal{V}$ *is the open coderivation* $\lfloor\mathcal{D}\rfloor_\mathcal{V} = \mathcal{D}\{\mathsf{hyp}/\nu\}_{\nu\in\mathcal{V}}$*. If* $\mathcal{D}$ *and* $\mathcal{D}'$ *are two open coderivations, then we say that* $\mathcal{D}$ *is an* ***approximation*** *of* $\mathcal{D}'$ *(noted* $\mathcal{D} \preceq \mathcal{D}'$*) iff* $\mathcal{D} = \lfloor\mathcal{D}'\rfloor_\mathcal{V}$ *for some* $\mathcal{V} \subseteq \{1,2\}^*$*. An approximation is* ***finite*** *if it is an open derivation.*

Note that $\mathcal{D}$ and $\lfloor\mathcal{D}\rfloor_\mathcal{V}$ (and hence $\mathcal{D}'$ if $\mathcal{D} \preceq \mathcal{D}'$) have the same conclusion.

▶ **Proposition 24.** *For any sequent* Γ*, the poset* $(\mathsf{oPLL}^\infty(\Gamma), \preceq)$ *is a Scott-domain with least element the open derivation* hyp *and with maximal elements the coderivations (in* PLL^∞*) with conclusion* Γ*. The compact elements are precisely the open derivations in* $\mathsf{oPLL}^\infty(\Gamma)$*.*

[4] Previously introduced notions and definitions on coderivations extend to open coderivations in the obvious way, e.g. the global conditions Definitions 11 and 16 and the cut-elimination relation $\to_{\mathsf{cut}}$.

Cut-elimination steps essentially do not increase the size of open derivations, hence:

▶ **Lemma 25.** $\to_{\mathsf{cut}}$ *over open derivations is strongly normalizing and confluent.*

Progressing and finitely expandable coderivations can be approximated in a canonical way.

▶ **Proposition 26.** *If $\mathcal{D} \in \mathsf{pPLL}^{\infty}$ is finitely expandable, then there is a finite set $\mathcal{V} \subseteq \{1,2\}^*$ of nodes of $\mathcal{D}$ such that $\lfloor \mathcal{D} \rfloor_{\mathcal{V}}$ is a open derivation and each $v \in \mathcal{V}$ is the root of a* nwb *in $\mathcal{D}$.*

Proof. By Lemma 20, there is a set $\mathcal{V}$ of nodes of $\mathcal{D}$ such that: (i) each node in $\mathcal{V}$ is the root of a nwb, and (ii) any infinite branch of $\mathcal{D}$ contains a node in $\mathcal{V}$. Thus, $\lfloor \mathcal{D} \rfloor_{\mathcal{V}}$ must be finite by weak König's lemma, and so is $\mathcal{V}$. ◀

▶ **Definition 27.** *Let $\mathcal{D} \in \mathsf{pPLL}^{\infty}$ be finitely expandable. The* ***decomposition*** *of $\mathcal{D}$ is the (unique) set of nodes $\mathsf{border}(\mathcal{D}) = \{\nu_1, \dots, \nu_k\}$ with $k \in \mathbb{N}$ such that $\mathcal{D}_{\nu_i}$ is a* nwb *for all $i \in \{1, \dots, k\}$ and $\mathsf{base}(\mathcal{D}) \coloneqq \lfloor \mathcal{D} \rfloor_{\mathsf{border}(\mathcal{D})}$ is a minimal (w.r.t. $\preceq$) finite approximation.*

5.2 Domain-theoretic approach to continuous cut-elimination

In this subsection we define *maximal and continuous infinitary cut-elimination strategies* (mc-ices), special rewriting strategies that stepwise generate ω-chains approximating the cut-free version of an open coderivation. In other words, a mc-ices computes a (Scott-)continuous function from open coderivations to cut-free open coderivations. Then, we introduce the *height-by-height* mc-ices, a notable example of mc-ices that will be used for our results, and we show that any two mc-icess compute the same (Scott-)continuous function.

In what follows, σ denotes a countable sequence of coderivations, and $\sigma(i)$ denotes the $i+1$-th coderivation in σ. We denote the length of a sequence σ by $\ell(\sigma) \leq \omega$.

▶ **Definition 28.** *An* ***infinitary cut elimination strategy*** *(or* ices *for short) is a family $\sigma = \{\sigma_{\mathcal{D}}\}_{\mathcal{D} \in \mathsf{oPLL}^{\infty}}$ where, for all $\mathcal{D} \in \mathsf{oPLL}^{\infty}$, $\sigma_{\mathcal{D}}$ is a sequence of open coderivations such that $\sigma_{\mathcal{D}}(0) = \mathcal{D}$ and $\sigma_{\mathcal{D}}(i) \to_{\mathsf{cut}} \sigma_{\mathcal{D}}(i+1)$ for all $0 \leq i < \ell(\sigma_{\mathcal{D}})$. Given a* ices *$\sigma$, we define the function $f_\sigma \colon \mathsf{oPLL}^{\infty}(\Gamma) \to \mathsf{oPLL}^{\infty}(\Gamma)$ as $f_\sigma(\mathcal{D}) \coloneqq \bigsqcup_{i=0}^{\ell(\sigma_{\mathcal{D}})} \mathsf{cf}(\sigma_{\mathcal{D}}(i))$ where $\mathsf{cf}(\mathcal{D}_i)$ is the greatest cut-free approximation of $\mathcal{D}_i$ (w.r.t. $\preceq$)[5]. An* ices *σ is a* mc-ices *if it is:*

- ***maximal****: $\sigma_{\mathcal{D}}(\ell(\sigma_{\mathcal{D}}))$ is normal for any open derivation $\mathcal{D}$ ($\ell(\sigma_{\mathcal{D}}) < \omega$ by Lemma 25);*
- ***(Scott)-continuous****: f_σ is Scott-continuous.*

Roughly, a maximal ices is a ices that applies cut-elimination steps to open derivations (i.e., finite approximations) until a normal (possibly cut-free) open derivation is reached.

The following property states that all mc-icess induce the same continuous function, an easy consequence of Lemma 25 and continuity.

▶ **Proposition 29.** *If σ and σ' are two* mc-ices*s, then $f_\sigma = f_{\sigma'}$.*

Therefore, we define a specific mc-ices we use in our proofs, where cut-elimination steps are applied in a deterministic way to the minimal reducible cut-rules.

▶ **Definition 30.** *The* ***height-by-height*** ices *is defined as $\sigma^{\infty} = \{\sigma^{\infty}_{\mathcal{D}}\}_{\mathcal{D} \in \mathsf{oPLL}^{\infty}}$ where $\sigma^{\infty}_{\mathcal{D}}(0) = \mathcal{D}$ for each $\mathcal{D} \in \mathsf{oPLL}^{\infty}$, and $\sigma^{\infty}_{\mathcal{D}}(i+1)$ is the open coderivation obtained by applying a cut-elimination step to the leftmost reducible* cut*-rule with minimal height in $\sigma^{\infty}_{\mathcal{D}}(i)$.*

▶ **Proposition 31.** *The* ices *σ^{∞} is a* mc-ices*.*

---

[5] f_σ is well-defined, as $(\mathsf{cf}(\sigma_{\mathcal{D}}(i)))_{0 \leq i < \ell(\sigma_{\mathcal{D}})}$ is an ω-chain in oPLL^{∞} and so its sup exists by Proposition 24.

Proof. By definition, σ^∞ is continuous. It is also maximal since, by Lemma 25, any open derivation $\mathcal{D}$ normalizes in $n_\mathcal{D} \in \mathbb{N}$ steps; so, $\ell(\sigma^\infty_\mathcal{D}) = n_\mathcal{D}$ and $\sigma^\infty_\mathcal{D}(h_\mathcal{D})$ is normal. ◀

We conclude this section by providing the sketch of proof for the continuous cut-elimination theorem, the main contribution of this paper, establishing a productivity result and showing that continuous cut-elimination preserves all global conditions.

▶ **Theorem 32** (Continuous Cut-Elimination).
1. *If $\mathcal{D} \in \mathsf{wpPLL}^\infty$, then $f_{\sigma^\infty}(\mathcal{D}) \in \mathsf{PLL}^\infty$.*
2. *If $\mathcal{D} \in \mathsf{wpPLL}^\infty$ (resp. $\mathcal{D} \in \mathsf{pPLL}^\infty$), then so is $f_{\sigma^\infty}(\mathcal{D})$.*
3. *If $\mathcal{D} \in \mathsf{wpPLL}^\infty$ is finitely expandable, then so is $f_{\sigma^\infty}(\mathcal{D})$.*
4. *If $\mathcal{D} \in \mathsf{wrPLL}^\infty$ (resp. $\mathcal{D} \in \mathsf{rPLL}^\infty$), then so is $f_{\sigma^\infty}(\mathcal{D})$.*

Sketch of proof.
1. It suffices to prove that for any $h \geq 0$ there is $n_h \geq 0$ such that $\mathsf{cf}(\sigma^\infty_\mathcal{D}(n_h))$ has a hyp-free bar $\mathcal{V}_h$ of rules in $\{\mathsf{ax}, \mathbf{1}, \mathsf{c!p}\}$ of height greater than h. The existence of a starting bar for $\mathcal{D} = \sigma^\infty_\mathcal{D}(0)$ is ensured by weak-progressing condition. Then, we show how to define bars of greater height through cut-elimination. The key case is when a c!p-rule in the bar is eliminated by a c!p-vs-?b step, in which case we exploit Proposition 21 to find such a new bar. The crucial property to establish is that only finitely many refinements of a starting bar are needed to find the $\mathcal{V}_h$, which follows from the fact that, by weak-progressing condition, there is no branch of $\mathcal{D}$ that contains infinitely many consecutive ?b rules.
2. We prove the result for $\mathcal{D} \in \mathsf{pPLL}^\infty$ since the proof for $\mathcal{D} \in \mathsf{wpPLL}^\infty$ is similar. By the previous point, $f_{\sigma^\infty}(\mathcal{D}) \in \mathsf{PLL}^\infty$. By Proposition 21 $\sigma^\infty_\mathcal{D}(i)$ is progressing for all $i < \omega$. Therefore if $f_{\sigma^\infty}(\mathcal{D})$ contains a non-progressing branch $\mathcal{B}$, it must have been stepwise constructed by pushing upward a cut-rule in $\mathcal{D}$. We can track the occurrences of this cut-rule in $\sigma^\infty_\mathcal{D}$ to define a sequence $(\mathsf{r}_0, \mathsf{r}_1, \ldots, \mathsf{r}_n, \ldots)_{i \leq \omega}$ of cut-rules such that $\mathsf{r}_i \in \sigma^\infty_\mathcal{D}(i)$ and either $\mathsf{r}_i = \mathsf{r}_{i+1}$ or $\sigma^\infty_\mathcal{D}(i) \to_{\mathsf{cut}} \sigma^\infty_\mathcal{D}(i+1)$ by applying a cut-elimination step on r_i producing r_{i+1}. This sequence of cut rules must reduce infinitely many occurrences of a formula $?A^\perp$ (in a same ?-thread) with infinitely many occurrences of a $!A$ (in a same !-thread). That is, there are infinitely many cut-elimination step c!p-vs-c!p in the $\sigma^\infty_\mathcal{D}$ producing an infinite progressing !-thread in $\mathcal{B}$.
3. Similar to the previous point.
4. Akin to linear logic, we define the *depth* of a coderivation as the maximal number of nested nwb*s*, and we prove that the depth of progressing and finitely expandable coderivations is always finite. Moreover, by Proposition 26, a weak-progressing and infinitely expandable coderivation $\mathcal{D}$ can be decomposed to a nwb-free finite approximation $\mathsf{base}(\mathcal{D})$ and a series of nwb*s* $\mathfrak{S}_1, \ldots, \mathfrak{S}_k$ with smaller depth. Using this property we define by induction on the depth of $\mathcal{D}$ a maximal and *transfinite* ices reducing the calls of the nwb*s* orderly, that is, reducing the i-th call to a cut-free coderivation before reducing the $i+1$-th one. This transfinite ices has the advantage of making apparent the preservation of (weak) regularity under cut-elimination: leveraging on Remark 17, if we reduce a nwb with finite support (resp. a periodic nwb) via our transfinite ices, then we obtain in the limit a cut-free nwb with finite support (resp. a periodic nwb). We conclude by showing that this transfinite ices can be compressed to a (ω-long) mc-ices using methods studied in [32, 29]. ◀

By definition (as the sup of cut-free open coderivations) $f_{\sigma^\infty}(\mathcal{D})$ is cut-free. Each item of Theorem 32 say in particular that $f_{\sigma^\infty}(\mathcal{D})$ is hyp-free, which means that $f_{\sigma^\infty}(\mathcal{D})$ is obtained by eliminating *all* the cuts in $\mathcal{D}$. This may not be the case if $\mathcal{D}$ does not fulfill any of the global conditions in the hypotheses of Theorem 32: $f_{\sigma^\infty}(\mathcal{D})$ is still cut-free but may contain some "truncating" hyp that "prevented" eliminating some cut in $\mathcal{D}$, as in the example below.

$$\left\{\!\!\left\{ \mathsf{ax}\,\frac{}{A, A^\perp} \right\}\!\!\right\}_n = \{\, (x,x) \mid x \in \{\{A\}\} \,\} \qquad \left\{\!\!\left\{ \mathsf{cut}\,\frac{\overset{\mathcal{D}'}{\Gamma, A} \quad \overset{\mathcal{D}''}{\Delta, A^\perp}}{\Gamma, \Delta} \right\}\!\!\right\}_n = \left\{ (\vec{x}, \vec{y}) \;\middle|\; \exists z \in \{\{A\}\} \text{ s.t. } \begin{array}{c} (\vec{x}, z) \in \{\!\{\mathcal{D}'\}\!\}_{n-1} \\ \text{and} \\ (z, \vec{y}) \in \{\!\{\mathcal{D}''\}\!\}_{n-1} \end{array} \right\}$$

$$\left\{\!\!\left\{ \bot\,\frac{\overset{\mathcal{D}'}{\Gamma}}{\Gamma, \bot} \right\}\!\!\right\}_n = \{\, (\vec{x}, *) \mid \vec{x} \in \{\!\{\mathcal{D}'\}\!\}_{n-1} \,\} \qquad \left\{\!\!\left\{ \parr\,\frac{\overset{\mathcal{D}'}{\Gamma, A, B}}{\Gamma, A \parr B} \right\}\!\!\right\}_n = \{\, (\vec{x}, (y,z)) \mid (\vec{x}, y, z) \in \{\!\{\mathcal{D}'\}\!\}_{n-1} \,\}$$

$$\left\{\!\!\left\{ 1\,\frac{}{1} \right\}\!\!\right\}_n = \{*\} \qquad \left\{\!\!\left\{ \otimes\,\frac{\overset{\mathcal{D}'}{\Gamma, A} \quad \overset{\mathcal{D}''}{\Delta, B}}{\Gamma, \Delta, A \otimes B} \right\}\!\!\right\}_n = \left\{ (\vec{x}, \vec{y}, (x,y)) \;\middle|\; \begin{array}{c} (\vec{x}, x) \in \{\!\{\mathcal{D}'\}\!\}_{n-1} \\ \text{and} \\ (\vec{y}, y) \in \{\!\{\mathcal{D}''\}\!\}_{n-1} \end{array} \right\} \qquad \left\{\!\!\left\{ \mathsf{hyp}\,\frac{}{\Gamma} \right\}\!\!\right\}_n = \varnothing$$

$$\left\{\!\!\left\{ ?\mathsf{w}\,\frac{\overset{\mathcal{D}'}{\Gamma}}{\Gamma, ?A} \right\}\!\!\right\}_n = \{\, (\vec{x}, [\,]) \mid \vec{x} \in \{\!\{\mathcal{D}'\}\!\}_{n-1} \,\} \qquad \left\{\!\!\left\{ ?\mathsf{b}\,\frac{\overset{\mathcal{D}'}{\Gamma, A, ?A}}{\Gamma, ?A} \right\}\!\!\right\}_n = \{\, (\vec{x}, [y] + \mu) \mid (\vec{x}, y, \mu) \in \{\!\{\mathcal{D}'\}\!\}_{n-1} \,\}$$

$$\left\{\!\!\left\{ \mathsf{c!p}\,\frac{\overset{\mathcal{D}'}{\Gamma, A} \quad \overset{\mathcal{D}''}{?\Gamma, !A}}{?\Gamma, !A} \right\}\!\!\right\}_n = \left\{ (\vec{[\,]}, [\,]) \right\} \cup \left\{ ([x_1] + \mu_1, \ldots, [x_k] + \mu_k, [x] + \mu) \;\middle|\; \begin{array}{c} (x_1, \ldots, x_k, x) \in \{\!\{\mathcal{D}'\}\!\}_{n-1} \\ \text{and} \\ (\mu_1, \ldots, \mu_k, \mu) \in \{\!\{\mathcal{D}''\}\!\}_{n-1} \end{array} \right\}$$

Figure 11 Inductive definition of the set $\{\!\{\mathcal{D}\}\!\}_n$, for $n > 0$.

▶ **Example 33.** For any finite approximation $\mathcal{D}$ of the (non-weakly progressing, non-finitely expandable) open coderivation $\mathcal{D}_{\lightning}$, we have $f_{\sigma^\infty}(\mathcal{D}) = \mathsf{hyp}$, so $f_{\sigma^\infty}(\mathcal{D}_{\lightning}) = \mathsf{hyp}$ by continuity.

6 Relational semantics for non-wellfounded proofs

Here we define a denotational model for oPLL^∞ based on *relational semantics*, which interprets an open coderivation as the union of the interpretations of its finite approximations, as in [14]. We show that relational semantics is sound for oPLL^∞, but not for its extension with digging.

Relational semantics interprets exponential by finite multisets, denoted by brackets, e.g., $[x_1, \ldots, x_n]$; $+$ denotes the *multiset union*, $\mathcal{M}_f(X)$ denotes the set of finite multisets over a set X. To correctly define the semantics of a coderivation, we need to see sequents as *finite sequence* of formulas (taking their order into account), which means that we have to add an *exchange* rule to oPLL^∞ to swap the order of two consecutive formulas in a sequent.

▶ **Definition 34.** *We associate with each formula A a **set** $\{\{A\}\}$ defined as follows:*

$$\{\{X\}\} := D_X \quad \{\{1\}\} := \{*\} \quad \{\{A \otimes B\}\} := \{\{A\}\} \times \{\{B\}\} \quad \{\{!A\}\} := \mathcal{M}_f(\{\{A\}\}) \quad \{\{A^\perp\}\} := \{\{A\}\}$$

where D_X is an arbitrary set. For a sequent $\Gamma = A_1, \ldots, A_n$, we set $\{\{\Gamma\}\} := \{\{A_1 \parr \cdots \parr A_n\}\}$.

Given $\mathcal{D} \in \mathsf{PLL} \cup \mathsf{oPLL}^\infty$ with conclusion Γ, we set $\{\{\mathcal{D}\}\} := \bigcup_{n \geq 0} \{\{\mathcal{D}\}\}_n \subseteq \{\{\Gamma\}\}$, where $\{\{\mathcal{D}\}\}_0 = \varnothing$ and, for all $i \in \mathbb{N} \setminus \{0\}$, $\{\{\mathcal{D}\}\}_i$ is defined inductively according to Figure 11.

▶ **Example 35.** For the coderivations $\mathcal{D}_{\lightning}$ and $\mathcal{D}_?$ in Figure 6, $\{\{\mathcal{D}_{\lightning}\}\} = \{\{\mathcal{D}_?\}\} = \varnothing$. For the derivations $\underline{\mathbf{0}}$ and $\underline{\mathbf{1}}$ in Figure 2, $\{\{\underline{\mathbf{0}}\}\} = \{([\,], (x,x)) \mid x \in D_X\}$ and $\{\{\underline{\mathbf{1}}\}\} = \{([(x,y)], (x,y)) \mid x, y \in D_X\}$. For the coderivation $\mathsf{c!p}_{(\underline{i_0}, \ldots, \underline{i_n}, \ldots)}$ in Example 10 (with $i_j \in \{\mathbf{0}, \mathbf{1}\}$ for all $j \in \mathbb{N}$), $\{\{\mathsf{c!p}_{(\underline{i_0}, \ldots, \underline{i_n}, \ldots)}\}\} = \{[\,]\} \cup \left\{[x_{i_0}, \ldots, x_{i_n}] \in \mathcal{M}_f(\{\{\mathbf{N}\}\}) \mid n \in \mathbb{N},\ x_{i_j} \in \{\{\underline{i_j}\}\}\ \forall 0 \leq j \leq n\right\}$.

By inspecting the cut-elimination steps and by continuity, we can prove the soundness of relational semantics with respect to cut-elimination (Theorem 37), thanks to the fact the interpretation of a coderivation is the union the interpretations of its finite approximation.

▶ **Lemma 36.** *Let $\mathcal{D} \in \mathsf{oPLL}^\infty$. Then, $\{\{\mathcal{D}\}\} = \bigcup_{\mathcal{D}' \in \mathcal{K}(\mathcal{D})} \{\{\mathcal{D}'\}\}$.*

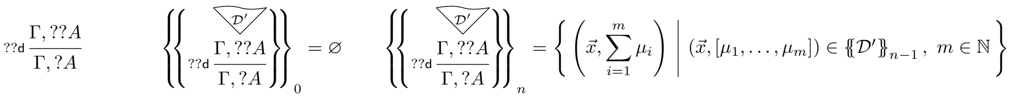


Figure 12 The rule ??d and its interpretation in the relational semantics ($n > 0$).

▶ **Theorem 37** (Soundness). **1.** *Let* $\mathcal{D} \in \mathsf{oPLL}^\infty$*. If* $\mathcal{D} \to_{\mathsf{cut}} \mathcal{D}'$*, then* $\{\{\mathcal{D}\}\} = \{\{\mathcal{D}'\}\}$*.*
2. *Let* $\mathcal{D} \in \mathsf{oPLL}^\infty$*. If* σ *is a* mc-ices*, then* $\{\{\mathcal{D}\}\} = \{\{f_\sigma(\mathcal{D})\}\}$*.*

By Theorem 37 and since cut-free coderivations have non-empty semantics, we have:

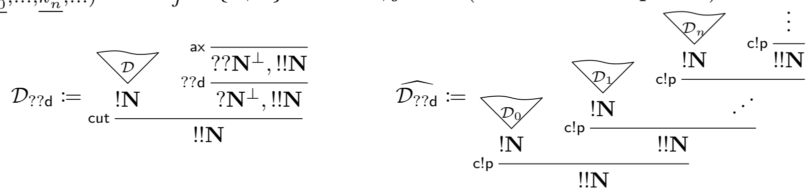
▶ **Corollary 38.** *Let* $\mathcal{D} \in \mathsf{wpPLL}^\infty$*. Then* $\{\{\mathcal{D}\}\} \neq \varnothing$*.*

We define the set of rules $\mathsf{MELL}^\infty := \mathsf{PLL}^\infty \cup \{??\mathsf{d}\}$ where the rule $??\mathsf{d}$ (**digging**) is defined in Figure 12. We also denote by MELL^∞ the set of coderivations over the rules in MELL^∞. Relational semantics is naturally extended to MELL^∞ as shown in Figure 12.

The proof system MELL^∞ can be seen as a non-wellfounded version of MELL. We show that, as opposed to several fragments of PLL^∞, in any good fragment of MELL^∞ with digging, cut-elimination cannot reduce to cut-free coderivations preserving the relational semantics.

▶ **Theorem 39.** *Let* $\mathsf{X} \subseteq \mathsf{MELL}^\infty$ *contain non-wellfounded coderivations with* $??\mathsf{d}$*. Let* $\to_{\mathsf{cut}+}$ *be a cut-elimination relation on* X *containing* $\to_{\mathsf{cut}}$ *in Figures 3, 5, and 9 and reducing every coderivation in* X *to a cut-free one. Then,* $\to_{\mathsf{cut}+}$ *does not preserve relational semantics.*

Proof. Consider the coderivations $\mathcal{D}_{??\mathsf{d}}$ and $\widehat{\mathcal{D}_{??\mathsf{d}}}$ below, where $\mathcal{D} = \mathsf{c!p}_{(\underline{\mathbf{0}},\underline{\mathbf{1}},\underline{\mathbf{0}},\underline{\mathbf{1}},\dots)}$, and $\mathcal{D}_i = \mathsf{c!p}_{(\underline{k^i_0},\dots,\underline{k^i_n},\dots)}$ and $k^i_j \in \{\mathbf{0},\mathbf{1}\}$ for all $i,j \in \mathbb{N}$ (see also Example 10).



(5)

Coderivations $\widehat{\mathcal{D}_{??\mathsf{d}}}$ are the only cut-free ones with conclusion $!!\mathbf{N}$. Therefore, for whatever definition of the cut-elimination steps concerning $??\mathsf{d}$, necessarily $\mathcal{D}_{??\mathsf{d}}$ will reduce to $\widehat{\mathcal{D}_{??\mathsf{d}}}$.

Let $\hat{\mathbf{0}}$ be the only element of $\{\{\underline{\mathbf{0}}\}\}$, and $\hat{\mathbf{1}}$ be any element of $\{\{\underline{\mathbf{1}}\}\}$ (see Example 35). Note that $\hat{\mathbf{0}} \neq \hat{\mathbf{1}}$. It is easy to verify that $[[\hat{\mathbf{0}}],[\hat{\mathbf{0}}]],[[\hat{\mathbf{1}}],[\hat{\mathbf{1}}]] \notin \{\{\mathcal{D}_{??\mathsf{d}}\}\}$, since $[\hat{\mathbf{0}},\hat{\mathbf{0}}],[\hat{\mathbf{1}},\hat{\mathbf{1}}] \notin \{\{\mathcal{D}\}\}$ (see Example 35). Concerning $\{\{\widehat{\mathcal{D}_{??\mathsf{d}}}\}\}$, we notice that, since $k^0_0, k^1_0, k^2_0 \in \{\mathbf{0},\mathbf{1}\}$, either $k^0_0 = k^1_0$ or $k^1_0 = k^2_0$ or $k^2_0 = k^0_0$. In the first case, we have $[[k^0_0],[k^1_0]] \in \{\{\widehat{\mathcal{D}_{??\mathsf{d}}}\}\}$, in the second case we have $[[k^1_0],[k^2_0]] \in \{\{\widehat{\mathcal{D}_{??\mathsf{d}}}\}\}$, and in the last case we have $[[k^2_0],[k^0_0]] \in \{\{\widehat{\mathcal{D}_{??\mathsf{d}}}\}\}$. ◀

7 Conclusion and future work

For future research, we envisage extending our contributions in many directions. First, our notion of finite approximation seems intimately related with that of Taylor expansion from *differential linear logic* (DiLL) [12], where the rule hyp (quite like the rule 0 from DiLL) serves to model approximations of *boxes*. This connection with Taylor expansions becomes even more apparent in Mazza's original systems for parsimonious logic [22, 23], which comprise co-absorption and co-weakening rules typical of DiLL. These considerations deserve further investigations. Secondly, building on a series of recent works in *Cyclic Implicit Complexity*, i.e., implicit computational complexity in the setting of circular and non-wellfounded proof theory [8, 7], we are currently working on second-order extensions of wrPLL^∞ and rPLL^∞ to characterize the complexity classes $\mathbf{P}/\mathsf{poly}$ and $\mathbf{P}$ (see [21]). These results would reformulate in a non-wellfounded setting the characterization of $\mathbf{P}/\mathsf{poly}$ presented in [23].

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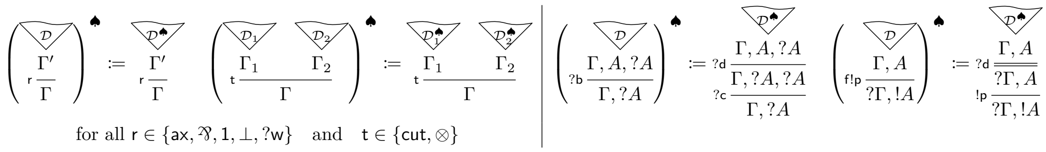


Figure 13 Translation $(\cdot)^{\spadesuit}$ from PLL to MELL.

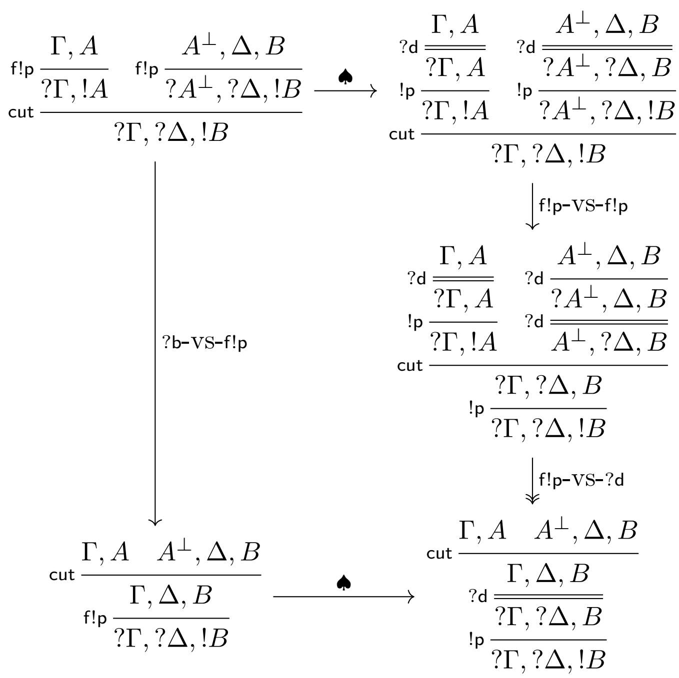


Figure 14 Commutation of the ?b-vs-f!p step and $(\cdot)^{\spadesuit}$.

A Appendix of Section 3

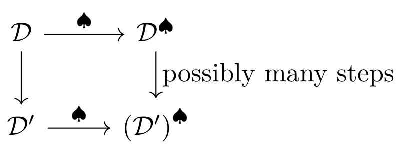
▶ **Theorem 5.** *For every $\mathcal{D} \in \mathsf{PLL}$, there is a cut-free $\mathcal{D}' \in \mathsf{PLL}$ such that $\mathcal{D} \to^*_{\mathsf{cut}} \mathcal{D}'$.*

Proof. We recall the sequent calculus for (propositional) *multiplicative exponential linear logic* $\mathsf{MELL} = \{\mathsf{ax}, \otimes, ⅋, 1, \bot, \mathsf{cut}, \mathsf{!p}, \mathsf{?w}, \mathsf{?d}, \mathsf{?c}\}$ where the **promotion** (!p), **dereliction** (?d), **contraction** (?c) rules are defined as follows:

$$\mathsf{!p}\,\frac{?\Gamma, A}{?\Gamma, !A} \qquad \mathsf{?d}\,\frac{\Gamma, A}{\Gamma, ?A} \qquad \mathsf{?c}\,\frac{\Gamma, ?A, ?A}{\Gamma, ?A} \tag{6}$$

We also denote by MELL the set of derivations over the rules in MELL, and we map each derivation in $\mathcal{D} \in \mathsf{PLL}$ to a derivation in $(\mathcal{D})^{\spadesuit} \in \mathsf{MELL}$ $(\cdot)^{\spadesuit} : \mathsf{PLL} \to \mathsf{MELL}$ defined in Figure 13 by induction on derivations.

In order to prove that the following diagram commute,



Each cut-elimination step in PLL corresponds to a cut-elimination step in MELL except the ones in Figures 14 and 15, where a cut-elimination step in PLL can be simulated by a sequence of cut-elimination steps in MELL. In these Figures each macro-step denoted by $\twoheadrightarrow$ involves a unique step from Figures 4 and 5 (the one marked) and certain additional commutative cut-elimination steps of the following form below

$$\dfrac{\Gamma, A \qquad \dfrac{A^\perp, \Delta, B}{A^\perp, \Delta, ?B}\,?\mathsf{d}}{\Gamma, \Delta, ?B}\,\mathsf{cut} \to_{\mathsf{cut}} \dfrac{\dfrac{\Gamma, A \qquad A^\perp, \Delta, B}{\Gamma, \Delta, B}\,\mathsf{cut}}{\Gamma, \Delta, ?B}\,?\mathsf{d} \qquad \dfrac{\dfrac{\Gamma, A, ?B, ?B}{\Gamma, ?A, ?B, ?B}\,?\mathsf{d}}{\Gamma, ?A, ?B}\,?\mathsf{c} \to_{\mathsf{cut}} \dfrac{\dfrac{\Gamma, A, ?B, ?B}{\Gamma, A, ?B}\,?\mathsf{c}}{\Gamma, ?A, ?B}\,?\mathsf{d} \tag{7}$$

which push ?d down a cut and create an alternating chain of ?d and ?c (such additional steps are natural to consider since they involve rule permutations of independent rules and would appear whenever a cut-rule would interact with the ?-formula introduced by the ?d-rule). Thus, the derivation in MELL obtained by (standard and additional) cut-elimination from $\mathcal{D}^\spadesuit$ is exactly the translation $(\mathcal{D}')^\spadesuit$ of the derivation $\mathcal{D}'$ in PLL obtained after a cut-elimination step from $\mathcal{D}$. According to the definition of $(\cdot)^\spadesuit$, if $(\mathcal{D}')^\spadesuit$ is cut-free then so is $\mathcal{D}'$.

The termination of cut-elimination in MELL with this additional commutative step follows from the result in MELL [26]. Indeed, to the usual measure m that decreases after each standard cut-elimination step in MELL (and remains unchanged after each additional step in (7)), we can add the sum d of the heights of the ?d rules in a derivation, which decreases after each step in (7). Thus, the measure (m, d) with the lexicographical order decreases after each (standard or additional) cut-elimination step in MELL. ◀

B Proofs of Section 4

Akin to linear logic, the *depth* of a coderivation is the maximal number of nested nwbs.

▶ **Definition 40.** *Let $\mathcal{D} \in \mathsf{PLL}^\infty$. The **nesting level of a sequent occurrence** Γ in $\mathcal{D}$ is the number $\mathbf{nl}_\mathcal{D}(\Gamma)$ of nodes below it that are the root of a call of a nwb. The **nesting level of a formula (occurrence)** A in $\mathcal{D}$, noted $\mathbf{nl}_\mathcal{D}(A)$, is the nesting level of the sequent that contain that formula. The **nesting level of a rule** r in $\mathcal{D}$, noted $\mathbf{nl}_\mathcal{D}(\mathsf{r})$ (resp. **of a sub-coderivation** $\mathcal{D}'$ of $\mathcal{D}$, noted $\mathbf{nl}_\mathcal{D}(\mathcal{D}')$), is the nesting level of the conclusion of r (resp. conclusion of $\mathcal{D}'$).*

*The **depth of** $\mathcal{D}$ is $\mathbf{d}(\mathcal{D}) \coloneqq \sup_{\mathsf{r}\in\mathcal{D}}\{\mathbf{nl}_\mathcal{D}(\mathsf{r})\} \in \mathbb{N} \cup \{\infty\}$.*

▶ Remark 41. All calls of a nwb have the same nesting level. Moreover, each of the sequents of its main branch have nesting level 0.

Cut-elimination $\to_{\mathsf{cut}}$ on PLL^∞ enjoys the following property.

▶ **Lemma 42.** *Let $\mathcal{D}, \mathcal{D}' \in \mathsf{PLL}^\infty$. If $\mathcal{D} \to_{\mathsf{cut}} \mathcal{D}'$ then $\mathbf{d}(\mathcal{D}) \geq \mathbf{d}(\mathcal{D}')$.*

Proof. By inspection of the cut-elimination steps in Figures 3, 5, and 9. ◀

▶ **Lemma 43.** *If $\mathcal{D} \in \mathsf{pPLL}^\infty$ then $\mathbf{d}(\mathcal{D}) \in \mathbb{N}$.*

Proof. If $\mathcal{D}$ had infinite depth, there would exist an infinite branch that goes left at c!p infinitely often. This branch cannot contain a (progressing) !-thread. ◀

▶ **Proposition 19.**
1. *If $\mathcal{D} \in \mathsf{PLL}$ (resp. $\mathcal{D} \in \mathsf{nuPLL}$) with conclusion Γ, then $\mathcal{D}^\circ \in \mathsf{rPLL}^\infty$ (resp. $\mathcal{D}^\bullet \in \mathsf{wrPLL}^\infty$) with conclusion Γ, and every c!p in $\mathcal{D}^\circ$ (resp. $\mathcal{D}^\bullet$) belongs to a nwb.*
2. *If $\mathcal{D}' \in \mathsf{rPLL}^\infty$ (resp. $\mathcal{D}' \in \mathsf{wrPLL}^\infty$) and every c!p in $\mathcal{D}'$ belongs to a nwb, then there is $\mathcal{D} \in \mathsf{PLL}$ (resp. $\mathcal{D} \in \mathsf{nuPLL}$) such that $\mathcal{D}^\circ = \mathcal{D}'$ (resp. $\mathcal{D}^\bullet = \mathcal{D}'$).*

Proof.
1. By straightforward induction on $\mathcal{D} \in \mathsf{PLL}$ (resp. $\mathcal{D} \in \mathsf{nuPLL}$).
2. By Lemma 43, $\mathbf{d}(\mathcal{D}) \in \mathbb{N}$. We can then prove the statement by induction on $\mathbf{d}(\mathcal{D})$. ◀

▶ **Proposition 21.** *Cut elimination preserves weak-regularity, regularity and finite expandability. Therefore, if $\mathcal{D} \in \mathsf{X}$ with $\mathsf{X} \in \{\mathsf{rPLL}^\infty, \mathsf{wrPLL}^\infty\}$ and $\mathcal{D} \to_{\mathsf{cut}} \mathcal{D}'$, then also $\mathcal{D}' \in \mathsf{X}$.*

Proof. By inspection of the cut-elimination steps defined in Figures 3, 5, and 9. ◀

C Proofs of Section 5

▶ **Lemma 25.** *$\to_{\mathsf{cut}}$ over open derivations is strongly normalizing and confluent.*

Proof. For $\mathcal{D}$ an open derivation, let $\mathsf{C}(\mathcal{D})$ be the number of $\mathsf{c!p}$ in $\mathcal{D}$ and $\mathsf{H}(\mathcal{D})$ be the sum of the sizes of all subderivations of $\mathcal{D}$ whose root is the conclusion of a cut rule. If $\mathcal{D} \to_{\mathsf{cut}} \mathcal{D}'$ via:
- a commutative cut-elimination step, then $\mathsf{C}(\mathcal{D}) = \mathsf{C}(\mathcal{D}')$, $|\mathcal{D}| = |\mathcal{D}'|$ and $\mathsf{H}(\mathcal{D}) > \mathsf{H}(\mathcal{D}')$;
- a multiplicative cut-elimination (Figure 3), then $\mathsf{C}(\mathcal{D}) = \mathsf{C}(\mathcal{D}')$ and $|\mathcal{D}| > |\mathcal{D}'|$;
- an exponential cut-elimination step (Figure 9), then $\mathsf{C}(\mathcal{D}) > \mathsf{C}(\mathcal{D}')$.

Since the lexicographic order over the triples $(\mathsf{C}(\mathcal{D}), |\mathcal{D}|, \mathsf{H}(\mathcal{D})) \in \omega^3$ is wellfounded, we conclude that there is no infinite sequence $(\mathcal{D}_i)_{i\in\mathbb{N}}$ such that $\mathcal{D}_0 = \mathcal{D}$ and $\mathcal{D}_i \to_{\mathsf{cut}} \mathcal{D}_{i+1}$.

Finally, since cut-elimination $\to_{\mathsf{cut}}$ is strongly normalizing over open derivations and it is locally confluent by inspection of critical pairs, by Newman's lemma it is also confluent. ◀

▶ **Proposition 29.** *If σ and σ' are two $\mathsf{mc\text{-}ices}$s, then $f_\sigma = f_{\sigma'}$.*

Proof. For any open derivation $\mathcal{D}$, since σ and σ' are maximal, we have that $\sigma_{\mathcal{D}}(\ell(\sigma_{\mathcal{D}}))$ and $\sigma'_{\mathcal{D}}(\ell(\sigma'_{\mathcal{D}}))$ are normal, and so $\sigma_{\mathcal{D}}(\ell(\sigma_{\mathcal{D}})) = \sigma'_{\mathcal{D}}(\ell(\sigma'_{\mathcal{D}}))$ by Lemma 25. Hence:

$$f_\sigma(\mathcal{D}) = \mathsf{cf}(\sigma_{\mathcal{D}}(\ell(\sigma_{\mathcal{D}}))) = \mathsf{cf}(\sigma'_{\mathcal{D}}(\ell(\sigma'_{\mathcal{D}}))) = f_{\sigma'}(\mathcal{D})$$

Now, let $\mathcal{D}$ be an open coderivation, and let $F(\mathcal{D})$ be the set of its finite approximations. Since by Proposition 24 oPLL^∞ is a Scott-domain, it is also algebraic, so that we have $\mathcal{D} = \bigsqcup_{\mathcal{D}' \in F(\mathcal{D})} \mathcal{D}'$. By continuity of f_σ and $f_{\sigma'}$ we have: $f_\sigma(\mathcal{D}) = \bigsqcup_{\mathcal{D}' \in F(\mathcal{D})} f_\sigma(\mathcal{D}') = \bigsqcup_{\mathcal{D}' \in F(\mathcal{D})} f_{\sigma'}(\mathcal{D}') = f_{\sigma'}(\mathcal{D})$. ◀

▶ **Definition 44** ($\mathsf{c!p}$-chains)**.** *Let $\sigma = \{\sigma_{\mathcal{D}}\}_{\mathcal{D} \in \mathsf{oPLL}^\infty}$ be a ices and let $\mathcal{D} \in \mathsf{oPLL}^\infty$. For any i, we write $\mathsf{r}_i \rightsquigarrow \mathsf{r}_{i+1}$ if r_i is a $\mathsf{c!p}$ rule in $\sigma_{\mathcal{D}}(i)$, r_{i+1} is a $\mathsf{c!p}$ rule in $\sigma_{\mathcal{D}}(i+1)$, and $\sigma_{\mathcal{D}}(i) \to_{\mathsf{cut}} \sigma_{\mathcal{D}}(i+1)$ is applied to a cut rule immediately below r_i and produces r_{i+1}. A $\mathsf{c!p}$-**chain** in $\sigma_{\mathcal{D}}$ is any sequence of $\mathsf{c!p}$ rules $(\mathsf{r}_i)_{i<\alpha}$ with $\alpha \leq \ell(\sigma_{\mathcal{D}})$ such that:*
- *for all $i \geq 0$, r_i is in $\sigma_{\mathcal{D}}(i)$*
- *either $\mathsf{r}_i = \mathsf{r}_{i+1}$ or $\mathsf{r}_i \rightsquigarrow \mathsf{r}_{i+1}$.*

▶ Remark 45. Let $\sigma = \{\sigma_{\mathcal{D}}\}_{\mathcal{D} \in \mathsf{oPLL}^\infty}$ be a ices. If r is a $\mathsf{c!p}$ rule in $\mathcal{D}$, then there is a *unique* maximal $\mathsf{c!p}$-chain $(\mathsf{r}_i)_{i<\alpha}$ in $\sigma_{\mathcal{D}}$ with ($\alpha \leq \ell(\sigma_{\mathcal{D}})$ and) $\mathsf{r} = \mathsf{r}_0$.

The following lemma establishes a productivity result for the height-by-height $\mathsf{mc\text{-}ices}$.

▶ **Lemma 46.** *If $\mathcal{D} \in \mathsf{wpPLL}^\infty$, then $f_{\sigma^\infty}(\mathcal{D}) \in \mathsf{PLL}^\infty$.*

Proof. Let $\mathcal{D}$ be a weakly progressing coderivation. Since $\mathcal{D}$ is by assumption hyp-free and no cut-elimination rule introduces hyp, we can assume $\ell(\sigma^\infty_{\mathcal{D}}) = \omega$. In what follows, we shorten $\sigma^\infty_{\mathcal{D}}(i)$ with $\mathcal{D}_i$, so $\mathcal{D}_0 = \mathcal{D}$. We show a stronger statement: for any $h \geq 0$ there is a $n_h \geq 0$ such that $\mathsf{cf}(\mathcal{D}_{n_h})$ has a hyp-free bar $\mathcal{V}_h$ of height greater than h. By definition, this will allow us to conclude that $f_{\sigma^\infty}(\mathcal{D}) = \bigsqcup_i \mathsf{cf}(\mathcal{D}_i)$ is hyp-free.

Let $h \geq 0$. We define a procedure computing $\mathcal{V}_h$ divided into rounds, where at the j-th round we compute $\mathcal{V}_h^j$. At round 0 we set $\mathcal{V}_h^0$ to be a bar across $\mathcal{D}$ with height greater than h containing only rules in $\{\mathsf{ax}, \mathbf{1}, \mathsf{c!p}\}$ (such a bar exists by weakly progressing). At the j-th round with $j > 0$, the procedure constructs $\mathcal{V}_h^j$ from $\mathcal{V}_h^{j-1}$. It analyses the first node of $\mathcal{V}_h^{j-1}$ that has not been considered in previous rounds (giving priority to nodes with highest prefix order)[6]. Let r^j be such a node. We only consider the case where r^j is a $\mathsf{c!p}$ rule. We consider the $\mathsf{c!p}$-chain $(\mathsf{r}_i^j)_i$ such that $\mathsf{r}^j = \mathsf{r}_0^j$ (which is unique by Remark 45). If there is a least i_0 such that $\mathsf{r}_{i_0}^j \rightsquigarrow \mathsf{r}_{i_0+1}^j$ (so that $\mathsf{r}_{i_0+1}^j$ is produced by applying a principal cut-elimination step to $\mathsf{r}_{i_0}^j$, and $\mathsf{r}_i^j = \mathsf{r}_{i+1}^j$ for all $i < i_0$), then we have three cases:

- If the cut-elimination step has shape $\mathsf{c!p}$-vs-$\mathsf{c!p}$ then we set $\mathcal{V}_h^j \coloneqq (\mathcal{V}_h^{j-1} \setminus \{\mathsf{r}^j\}) \cup \{\mathsf{r}_{i_0+1}^j\}$ and we move to the next round.
- If the cut-elimination step has shape $\mathsf{c!p}$-vs-$\mathsf{?w}$ then we set $\mathcal{V}_h^j \coloneqq \mathcal{V}_h^{j-1} \setminus \{\mathsf{r}^j\}$ and we move to the next round.
- Otherwise, the cut-elimination step has shape $\mathsf{c!p}$-vs-$\mathsf{?b}$. Let $\mathcal{D}'$ be the coderivation of $\sigma_{\mathcal{D}}$ containing the rule $\mathsf{r}_{i_0}^j$, let ν be the node of $\mathcal{D}'$ that is conclusion of $\mathsf{r}_{i_0}^j$, and let $\mathcal{U}_h^j$ be a suitable bar of $\mathcal{D}'_\nu$ at height > 0 containing only rules in $\{\mathsf{ax}, \mathbf{1}, \mathsf{c!p}\}$. This bar exists by weakly progressing of $\mathcal{D}$ and the fact that weak progressing is preserved under finite cut-elimination by Proposition 21. We set $\mathcal{V}_h^j \coloneqq (\mathcal{V}_h^{j-1} \setminus \{\mathsf{r}^j\}) \cup \mathcal{U}_h^j$ and we move to the next round.

If no such such $\mathsf{r}_{i_0}^j \rightsquigarrow \mathsf{r}_{i_0+1}^j$ exists (so $\mathsf{r}_i^j = \mathsf{r}_{i+1}^j$ for all i) we move to the next round.

By construction, if the procedure terminates, it computes the set of nodes $\mathcal{V}_h$ such that, for some $k \geq 0$ sufficiently large, $\mathcal{V}_h$ defines a bar across any $\mathcal{D}_i$ in the sequence $\sigma_{\mathcal{D}}$ for all $i \geq k$. This means that there exists $n_h \geq k$ such that $\mathsf{cf}(\mathcal{D}_{n_h})$ contains that bar. So we have to show that the procedure terminates. Since bars are finite, this boils down to proving that there are only finitely many rounds. Suppose towards contradiction that this is not the case. This can only happen when there are infinitely many distinct $\mathsf{c!p}$ rules $(\mathsf{r}_i)_i$ in a branch $\mathcal{B}_j$ of $\mathcal{D}$ and infinitely many distinct $\mathsf{?b}$ rules $(\mathsf{r}'_i)_i$ in a branch $\mathcal{B}'_j$ of $\mathcal{D}$ such that in $\sigma_{\mathcal{D}}^\infty$:

1. each r_i is eventually cut with r'_i,
2. each r_i is never cut with a $\mathsf{c!p}$ rule.

Notice that the assumption that the rules in $(\mathsf{r}'_i)_i$ belong to the same branch $\mathcal{B}'_j$ causes no loss of generality, since the height-by-height $\mathsf{mc\text{-}ices}$ reduces the cut r_i-vs-r'_i before any other cut above these rules. By Item 1 $\mathcal{B}'_j$ is infinite, and by Item 2 it is eventually $\mathsf{c!p}$-free, contradicting weakly progressing of $\mathcal{D}$. ◀

The following notion is the analogue of *(multi)cut reduction sequences* from [3].

▶ **Definition 47** (Cut-chains). *Let $\sigma = \{\sigma_{\mathcal{D}}\}_{\mathcal{D} \in \mathsf{oPLL}^\infty}$ be a* ices *and let $\mathcal{D} \in \mathsf{oPLL}^\infty$. For any i, we write $\mathsf{r}_i \mapsto \mathsf{r}_{i+1}$ if r_i is a* cut *rule in $\sigma_{\mathcal{D}}(i)$, r_{i+1} is a* cut *rule in $\sigma_{\mathcal{D}}(i+1)$, and $\sigma_{\mathcal{D}}(i) \to_{\mathsf{cut}} \sigma_{\mathcal{D}}(i+1)$ is applied to r_i producing r_{i+1}. A* cut-***chain*** *in $\sigma_{\mathcal{D}}$ is any sequence of* cut *rules $(\mathsf{r}_i)_{i<\alpha}$ with $\alpha \leq \ell(\sigma_{\mathcal{D}})$ such that:*

- *for all $i \geq 0$, r_i is in $\sigma_{\mathcal{D}}(i)$*
- *either $\mathsf{r}_i = \mathsf{r}_{i+1}$ or $\mathsf{r}_i \mapsto \mathsf{r}_{i+1}$.*

▶ Remark 48. Let $\sigma = \{\sigma_{\mathcal{D}}\}_{\mathcal{D} \in \mathsf{oPLL}^\infty}$ be a ices , and let $(\mathsf{r}_i)_i$ be an infinite cut chain in $\sigma_{\mathcal{D}}$ such that $(A_i, A_i^\perp)$ is the pair of cut formulas of r_i. There is $i_0 \geq 0$ such that, for all $i \geq i_0$, A_i is a !-formula (and $A_i^\perp$ is a ?-formula).

[6] More precisely, $\nu = a_0 \dots a_n < c_0 \dots c_m = \nu'$ with $a_i, c_i \in \{1, 2\}$ iff there is $i_0 \leq m$ such that $a_i \leq c_i$ for any $0 \leq i \leq i_0$ and $a_{i_0+1} < c_{i_0+1}$.

▶ Remark 49. Any branch $\mathcal{B}$ in a progressing coderivation $\mathcal{D}$ contains at most (and hence exactly) one progressing !-thread. As a consequence, any infinite !-thread τ of a branch $\mathcal{B}$ in a progressing coderivation $\mathcal{D}$ must be progressing. Indeed, let τ and τ' be two infinite !-threads, and let us show that $\tau = \tau'$. Since $\mathcal{B}$ is progressing, it contains infinitely many $\mathsf{c!p}$ rules $(\mathsf{r}_i)_i$, so that there exists $n \geq 0$ such that both τ and τ' contain formulas below r_i. Since the conclusion of r_i has exactly one !-formula and τ is infinite, both τ and τ' must contains that formula, so that $\tau = \tau'$ by maximality of !-threads.

▶ **Lemma 50.**

1. *If $\mathcal{D} \in \mathsf{wpPLL}^\infty$ (resp. $\mathcal{D} \in \mathsf{pPLL}^\infty$), then so is $f_{\sigma^\infty}(\mathcal{D})$.*
2. *If $\mathcal{D} \in \mathsf{wpPLL}^\infty$ is finitely expandable, then so is $f_{\sigma^\infty}(\mathcal{D})$.*

Proof. Let us prove Item 1. Let $\mathcal{D}$ be a progressing open coderivation, and let us shorten $\sigma^\infty_\mathcal{D}(i)$ with $\mathcal{D}_i$, so $\mathcal{D}_0 = \mathcal{D}$. By Proposition 21 we can assume that $\ell(\sigma^\infty_\mathcal{D}) = \omega$.

We want to show that for any infinite cut-chain $(\mathsf{r}_i)_{i<\omega}$ in $\sigma^\infty_\mathcal{D}$ such that:

(I) r_0 is a cut rule with minimal height in $\mathcal{D}$

(II) $\pi(\mathsf{r}_i) = a_0 a_1 \ldots a_{n_i}$ is the address of r_i in $\mathcal{D}_i$ (with $n_i \leq n_{i+1}$),

there exists $0 \leq k_0 \leq n_0$ and an infinite family $\tau^* := (C_i)_{k_0 \leq i}$ of occurrences of a !-formula satisfying the following properties:

a $\tau^*_i := (C_j)_{k_0 \leq j \leq n_i}$ is a !-thread in $\pi(\mathsf{r}_i)$

b for any $m \geq 0$ there is i such that τ^*_i has m progressing points.

Notice that the property above allows us to conclude. Indeed, let $\mathcal{B}$ be an infinite branch of $f_{\sigma^\infty}(\mathcal{D})$. If $\mathcal{B}$ is in some $\mathcal{D}_i$, then it is progressing by Proposition 21. Otherwise, there exists an infinite cut-chain $(\mathsf{r}_i)_{i<\omega}$ in $\sigma^\infty_\mathcal{D}$ satisfying Item a, Item b and $\mathcal{B} = \{\pi(\mathsf{r}_i) \mid i \geq 0\}$. By Item a and Item b there is an infinite family $(C_i)_{n_0 \leq i}$ of occurrences of a !-formula that defines a progressing !-thread of $\mathcal{B}$.

So, let $(\mathsf{r}_i)_{i<\omega}$ be a cut-chain with minimal height such that:

- the premises of r_i are conclusions of the rules r'_i and r''_i
- $(A_i, A_i^\perp)$ are the cut formulas of r_i
- $\pi(\mathsf{r}_i) = a_0 a_1 \ldots a_{n_i}$ is the address of r_i in $\mathcal{D}_i$

By Remark 48, we can assume w.l.o.g. that $A_i = !B$ and $A_i^\perp = ?B^\perp$. It is easy to see that $\tau := (A_i)_i$ is an infinite !-thread of some branch $\mathcal{B}'$ of $\mathcal{D}$ and that $\tau' := (A_i^\perp)$ is an infinite ?-thread of some branch $\mathcal{B}''$ in $\mathcal{D}$. Moreover, by Remark 49 and by progressing criterion of $\mathcal{D}$, τ is progressing. This means that there are infinitely many i such that $\mathsf{r}'_i = \mathsf{r}''_i = \mathsf{c!p}$ (so that A_i is the principal !-formula of r'_i and $A_{i+1}^\perp$ is an auxiliary ?-formula of r''_i) and $\mathsf{r}_i \mapsto \mathsf{r}_{i+1}$. Let $\tau'' := (C_i)_i$ be the progressing !-thread of $\mathcal{B}''$. Since r_0 is a cut with minimal height, and the minimal height cut rules never decreases during cut-elimination, all cuts r_i in the cut chain have minimal height. This means that the first formula of τ'', i.e., C_0, is not a cut formula, and so it is in the end-sequent of $\mathcal{D}$. It is easy to see that the cut-elimination rules never affect τ'' (and its progressing points) while pushing upward the cut rules. This means that we can construct τ'' satisfying the properties Item a and Item b.

Let us now prove Item 2. Since $f_{\sigma^\infty}(\mathcal{D})$ is cut-free we only have to show that all of its infinite branches have only finitely many $\mathsf{?b}$ rules each. Let $\mathcal{D}$ be a finitely expandable open coderivation, and let us shorten $\sigma^\infty_\mathcal{D}(i)$ with $\mathcal{D}_i$, so $\mathcal{D}_0 = \mathcal{D}$. By Proposition 21 we can assume that $\ell(\sigma^\infty_\mathcal{D}) = \omega$.

We want to show that for any infinite cut-chain $(\mathsf{r}_i)_{i<\omega}$ in $\sigma^\infty_\mathcal{D}$ such that:

(I) r_0 is a cut rule with minimal height in $\mathcal{D}$

(II) $\pi(\mathsf{r}_i) = a_0 a_1 \ldots a_{n_i}$ is the address of r_i in $\mathcal{D}_i$ (with $n_i \leq n_{i+1}$), the branch $\mathcal{B} = \{\pi(\mathsf{r}_i) \mid i \geq 0\}$ of $f(\mathcal{D})$ has only finitely many distinct ?b rules. Note that the property above allows us to conclude. Indeed, let $\mathcal{B}$ be an infinite branch of $f_{\sigma^\infty}(\mathcal{D})$. If $\mathcal{B}$ is in some $\mathcal{D}_i$, then it is finitely expandable by Proposition 21. Otherwise, there is an infinite cut-chain $(\mathsf{r}_i)_{i<\omega}$ in $\sigma^\infty_{\mathcal{D}}$ such that $\mathcal{B} = \{\pi(\mathsf{r}_i) \mid i \geq 0\}$, so we are done by the above property.

Thus, let $(\mathsf{r}_i)_{i<\omega}$ be a cut-chain with minimal height such that:

- the premises of r_i are conclusions of the rules r'_i and r''_i
- $(A_i, A_i^\perp)$ are the cut formulas of r_i
- $\pi(\mathsf{r}_i) = a_0 a_1 \dots a_{n_i}$ is the address of r_i in $\mathcal{D}_i$

By Remark 48, we can assume w.l.o.g. that $A_i = !B$ and $A_i^\perp = ?B^\perp$. It is easy to see that $\tau := (A_i)_i$ is an infinite !-thread of some branch $\mathcal{B}'$ of $\mathcal{D}$ and that $\tau' := (A_i^\perp)$ is an infinite ?-thread of some branch $\mathcal{B}''$ in $\mathcal{D}$.

Let us suppose towards contradiction that $\mathcal{B}$ has infinitely many ?b rules. This means that, for any k there is n_k such that $\pi(\mathsf{r}_{n_k})$ contains k ?b rules. Since $\mathcal{D}$ is finitely expandable, there must be infinitely many $i \geq 0$ such that $\mathsf{r}_i \mapsto \mathsf{r}_{i+1}$ is obtained by applying the cut-elimination step c!p-vs-?b. But this would mean that the ?-thread τ' contains infinitely many principal rules for ?b rule, and so $\mathcal{B}''$ would contain infinitely many ?b rules, contradicting finite expandability of $\mathcal{D}$. ◀

▶ **Proposition 51.** *Let $\mathcal{D} \in \mathsf{wrPLL}^\infty$ (resp. rPLL^∞). Then $f_{\sigma^\infty}(\mathcal{D})$ admits a decomposition, and $\mathsf{base}(f_{\sigma^\infty}(\mathcal{D})) = \mathsf{base}(\sigma^\infty_{\mathcal{D}}(n))$ for some $n \geq 0$.*

Proof. By Lemma 46 and Lemma 50, $f_{\sigma^\infty}(\mathcal{D})$ is a cut free (hyp-free) coderivation and finitely expandable coderivation. By Proposition 26 $f_{\sigma^\infty}(\mathcal{D})$ admits a decomposition $\mathsf{border}(\mathcal{D}) = \{v_1, \dots, v_k\}$. By continuity, this means that there is $n \geq 0$ such that $\mathsf{base}(\sigma^\infty_{\mathcal{D}}(n)) = \mathsf{base}(f_{\sigma^\infty}(\mathcal{D}))$. Note that $\mathsf{base}(\sigma^\infty_{\mathcal{D}}(n))$ exists by Propositions 21 and 26. ◀

▶ **Lemma 52.** *If $\mathcal{D} \in \mathsf{wrPLL}^\infty$ (resp. $\mathcal{D} \in \mathsf{rPLL}^\infty$), then so is $f_{\sigma^\infty}(\mathcal{D})$.*

Proof. We define a maximal and transfinite ices $\sigma = \{\sigma_{\mathcal{D}}\}_{\mathcal{D} \in \mathsf{oPLL}^\infty}$ preserving weak regularity, and show that this strategy can be "compressed" to a mc-ices, $\sigma^* = \{\sigma^*_{\mathcal{D}}\}_{\mathcal{D} \in \mathsf{oPLL}^\infty}$, along the lines of [29]. We then conclude since $f_{\sigma^\infty} = f_{\sigma^*}$ by Proposition 29. So let $\mathcal{D} \in \mathsf{wrPLL}^\infty$. By induction on $d = \mathbf{d}(\mathcal{D})$ (which is finite by Lemma 43) we define $\sigma_{\mathcal{D}} = (\mathcal{D}_i)_i$ such that:

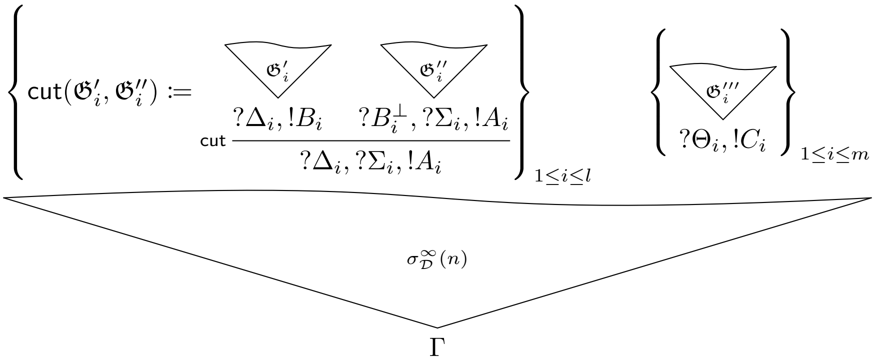
(a) For any limit ordinal $\lambda \leq \ell(\sigma_{\mathcal{D}})$:
  - **(i)** $\bigsqcup_{i<\lambda} \mathcal{D}'_i = \mathcal{D}_\lambda$ for some $\mathcal{D}'_i$ finite approximations of $\mathcal{D}_i$.
  - **(ii)** If h_i is the height of the cut reduced at the i-th step of $\sigma_{\mathcal{D}}$ then $\mathsf{lim}_{i<\lambda}(h_i) = \infty$.

(b) $\mathcal{D}_{\ell(\sigma_{\mathcal{D}})}$ is cut free

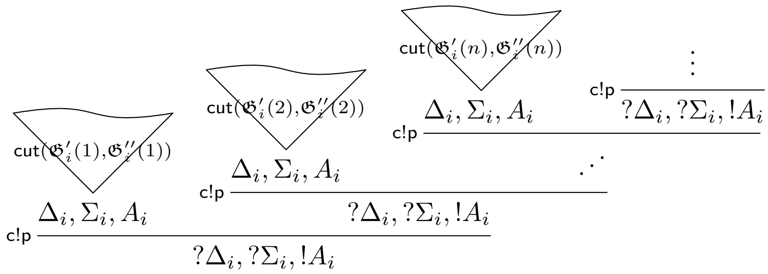
(c) $\mathcal{D}_{\ell(\sigma_{\mathcal{D}})}$ is weakly regular.

- If $d = 0$ then by Proposition 26 $\mathcal{D}$ is an open derivation, so that by Lemma 25 there is a maximal cut-elimination sequence that rewrites $\mathcal{D}$ to a normal open coderivation. In particular, the latter is also cut-free because $\mathcal{D}$ is hyp-free and so every cut can be eventually eliminated. We set $\sigma_{\mathcal{D}}$ to be such a cut-elimination sequence. By construction, $\sigma_{\mathcal{D}}$ satisfies Item ai-aii and Item b. Moreover, by Proposition 21 $\sigma_{\mathcal{D}}$ satisfies Item c
- If $d > 0$ then by Proposition 51 there is $n \geq 0$ such that $\mathsf{base}(f_{\sigma^\infty}(\mathcal{D})) = \mathsf{base}(\sigma^\infty_{\mathcal{D}}(n))$

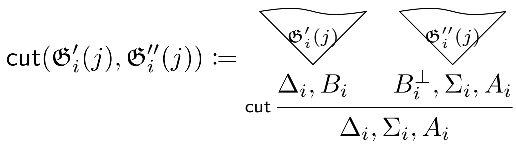
By construction $\sigma^{\infty}_{\mathcal{D}}(n)$ has the following structure:



for some nwbs $\mathfrak{G}'_i, \mathfrak{G}''_i, \mathfrak{G}'''_i$. For any $1 \leq i \leq l$, let σ_i be the mc-ices that applies only cut-elimination steps for c!p-vs-c!p and that rewrites $\mathsf{cut}(\mathfrak{G}'_i, \mathfrak{G}''_i)$ to the following coderivation:



where:



By induction hypothesis, for any $j \geq 0$ we have maximal transfinite icess $\sigma_{\mathsf{cut}(\mathfrak{G}'_i(j),\mathfrak{G}''_i(j))}$ and $\sigma_{\mathfrak{G}'''_i(j)}$ satisfying the hypothesis. Since $\mathcal{D}$ is weakly regular the sets of sequences $X_i \coloneqq \{\sigma_{\mathsf{cut}(\mathfrak{G}'_i(j),\mathfrak{G}''_i(j))} \mid j \geq 0\}$ and $Y_i \coloneqq \{\sigma_{\mathfrak{G}'''_i(j)} \mid j \geq 0\}$ can be assumed to be finite. We set:

$$\sigma_{\mathcal{D}} \coloneqq (\sigma^{\infty}_{\mathcal{D}}(i))_{0 \leq i \leq n} \cdot \prod_{i=1}^{m} \prod_{j=1}^{\infty} \sigma_{\mathfrak{G}'''_i(j)} \cdot \prod_{i=1}^{l} (\sigma_i \cdot \prod_{j=1}^{\infty} \sigma_{\mathsf{cut}(\mathfrak{G}'_i(j),\mathfrak{G}''_i(j))})$$

where $\sigma' \cdot \sigma''$ denotes the concatenation of two sequences σ' and σ''. Let us now show that $\sigma_{\mathcal{D}}$ satisfies Item ai-aii. This follows from the induction hypothesis and the construction of σ_i ($1 \leq i \leq l$). Notice, indeed, that the i-th element of σ_i is the application of a cut-elimination step to a cut with shape c!p-vs-c!p and with height i. Clearly, Item b is satisfied. Concerning Item c, since the sets of sequences X_i and Y_i are finite, using the induction hypothesis we have that if the sequences $\sigma_{X_i} \coloneqq \prod_{j=1}^{\infty} \sigma_{\mathfrak{G}'''_i(j)}$ and $\sigma_{Y_i} \coloneqq \sigma_i \cdot \prod_{j=1}^{\infty} \sigma_{\mathsf{cut}(\mathfrak{G}'_i(j),\mathfrak{G}''_i(j))}$ are applied to a weakly regular coderivation, their limit is a weakly regular coderivation. From this fact and Proposition 21 we can conclude that the limit of $\sigma_{\mathcal{D}}$ is weakly regular.

Now, let $\mathsf{lim}(\sigma_{\mathcal{D}})$ be the limit of $\sigma_{\mathcal{D}}$. We want to show by induction d that σ can be rewritten to a mc-ices σ^* such that $\mathsf{lim}(\sigma_{\mathcal{D}}) = f_{\sigma^*}(\mathcal{D})$.

- The case $d = 0$ follows by construction of σ.
- Let us suppose $d > 0$. By induction hypothesis we have $\sigma^*_{\mathsf{cut}(\mathfrak{G}'_i(j),\mathfrak{G}''_i(j))}$ and $\sigma^*_{\mathfrak{G}'''_i(j)}$ such that, for any $j \geq 0$:
  - $\lim\left(\sigma_{\mathsf{cut}(\mathfrak{G}'_i(j),\mathfrak{G}''_i(j))}\right) = f_{\sigma^*}(\mathsf{cut}(\mathfrak{G}'_i(j), \mathfrak{G}''_i(j)))$
  - $\lim\left(\sigma_{\mathfrak{G}'''_i(j)}\right) = f_{\sigma^*}(\mathfrak{G}'''_i(j))$.

  Let us now show that the sequences $\sigma_i \cdot \prod_{j=1}^{\infty} \sigma^*_{\mathsf{cut}(\mathfrak{G}'_i(j),\mathfrak{G}''_i(j))}$ can be rewritten to a sequence with length ω with the same limit and preserving conditions Item a-c. We notice that:
  - for any $j \neq j'$, cut-elimination steps in $\sigma^*_{\mathsf{cut}(\mathfrak{G}'_i(j),\mathfrak{G}''_i(j))}$ commute with cut-elimination steps in $\sigma^*_{\mathsf{cut}(\mathfrak{G}'_i(j'),\mathfrak{G}''_i(j'))}$
  - the $j+1$-th cut-elimination step of σ_i commutes with all cut-elimination steps in $\sigma(\mathsf{cut}(\mathfrak{G}'_i(j'), \mathfrak{G}''_i(j')))$ with $j' < j$.

  Since σ_i is has length ω, by the above observations, we define a sequence σ^*_i of length ω divided into stages, where each stage consists of a finite subsequence of reduction steps. At the n-th stage:
  - we apply the n-th cut-elimination step of σ_i
  - for any $1 \leq j \leq n$ we apply (the next available) $n + 1 - j$ steps of $\sigma^*_{\mathsf{cut}(\mathfrak{G}'_i(j),\mathfrak{G}''_i(j))}$.

  In a similar way, for any $1 \leq i \leq m$ the reduction sequence $\sigma^*_{\mathfrak{G}'''_i(j)}$ can be rewritten to a sequence $\sigma^{**}_i(\mathcal{D})$ of length $\leq \omega$ (preserving the limit and conditions Item a-c). We obtain a sequence of the following form:

$$(\sigma^{\infty}_{\mathcal{D}}(i))_{0\leq i\leq n} \cdot \prod_{i=1}^{m} \sigma^{**}_i \cdot \prod_{i=1}^{l} \sigma^*_i$$

  Since any cut-elimination step in σ^{**}_i commutes with any cut-elimination step in σ^*_i, we can rewrite the above sequence to a sequence $\sigma^*_{\mathcal{D}} = (\mathcal{D}_i)_i$ of length $\leq \omega$ with the same limit and preserving conditions Item a-c. By definition, to prove that $\lim(\sigma_{\mathcal{D}}) = f_{\sigma^*}(\mathcal{D})$ it suffices to show that $\lim(\sigma_{\mathcal{D}}) = \bigsqcup_i \mathsf{cf}(\mathcal{D}_i)$:
  - By Item ai we have $\lim(\sigma_{\mathcal{D}}) = \bigsqcup_i \mathcal{D}'_i$ for some $\mathcal{D}'_i$ approximations of $\mathcal{D}_i$ so that, by Item b, we have $\lim(\sigma_{\mathcal{D}}) = \bigsqcup_i \mathcal{D}'_i \preceq \bigsqcup_i \mathsf{cf}(\mathcal{D}_i)$.
  - By Item aii we have $\bigsqcup_i \mathsf{cf}(\mathcal{D}_i) \preceq \lim(\sigma_{\mathcal{D}})$.

This shows that $f_{\sigma^\infty}(\mathcal{D})$ is weakly regular if $\mathcal{D}$ is. Therefore, if $\mathcal{D} \in \mathsf{wrPLL}^\infty$ then $f_{\sigma^\infty}(\mathcal{D}) \in \mathsf{wrPLL}^\infty$ by Lemma 46 and Lemma 50.

Concerning preservation of regularity, we apply the same reasoning, checking that the ices preserves periodicity of nwbs. ◀

D Proofs of Section 6

▶ **Lemma 36.** *Let $\mathcal{D} \in \mathsf{oPLL}^\infty$. Then, $\{\{\mathcal{D}\}\} = \bigcup_{\mathcal{D}' \in \mathcal{K}(\mathcal{D})} \{\{\mathcal{D}'\}\}$.*

Proof. By Proposition 24, $\mathcal{D} = \bigsqcup_{\mathcal{D}' \in \mathcal{K}(\mathcal{D})} \mathcal{D}'$.

For the left-to-right inclusion, observe that for every $n \in \mathbb{N}$ there is $\mathcal{D}'_n \in \mathcal{K}(\mathcal{D})$ such that $\{\{\bigsqcup_{\mathcal{D}' \in \mathcal{K}(\mathcal{D})} \mathcal{D}'\}\}_n = \{\{\mathcal{D}'_n\}\} \subseteq \bigcup_{\mathcal{D}' \in \mathcal{K}(\mathcal{D})} \{\{\mathcal{D}'\}\}$. Therefore, by minimality of the union,

$$\{\{\mathcal{D}\}\} = \bigcup_{n\in\mathbb{N}} \{\{\mathcal{D}\}\}_n = \bigcup_{n\in\mathbb{N}} \{\{ \bigsqcup_{\mathcal{D}' \in \mathcal{K}(\mathcal{D})} \mathcal{D}'\}\}_n \subseteq \bigcup_{\mathcal{D}' \in \mathcal{K}(\mathcal{D})} \{\{\mathcal{D}'\}\}.$$

As for the converse inclusion, we have that $\mathcal{D}' \preceq \mathcal{D}''$ implies $\{\{\mathcal{D}'\}\} \subseteq \{\{\mathcal{D}''\}\}$. Hence, for all $\mathcal{D}' \in \mathcal{K}(\mathcal{D})$, since $\mathcal{D}' \preceq \bigsqcup_{\mathcal{D}' \in \mathcal{K}(\mathcal{D})} \mathcal{D}' = \mathcal{D}$, we have $\{\{\mathcal{D}'\}\} \subseteq \{\{\mathcal{D}\}\}$. By minimality of the union, $\bigcup_{\mathcal{D}' \in \mathcal{K}(\mathcal{D})} \{\{\mathcal{D}'\}\} \subseteq \{\{\mathcal{D}\}\}$. ◀

▶ **Theorem 37** (Soundness). **1.** *Let* $\mathcal{D} \in \mathsf{oPLL}^\infty$. *If* $\mathcal{D} \to_{\mathsf{cut}} \mathcal{D}'$, *then* $\{\{\mathcal{D}\}\} = \{\{\mathcal{D}'\}\}$.
2. *Let* $\mathcal{D} \in \mathsf{oPLL}^\infty$. *If* σ *is a* mc-ices, *then* $\{\{\mathcal{D}\}\} = \{\{f_\sigma(\mathcal{D})\}\}$.

Proof. **1.** By straightforward inspection of the cut-elimination steps for oPLL^∞.
2. By definition of mc-ices, for any $\mathcal{D}' \in \mathcal{K}(\mathcal{D})$ we have $\mathcal{D}' \to^*_{\mathsf{cut}} f_\sigma(\mathcal{D}')$, so $\{\{\mathcal{D}'\}\} = \{\{f_\sigma(\mathcal{D}')\}\}$ by Theorem 37.1. By Proposition 24, $\mathcal{D} = \bigsqcup_{\mathcal{D}' \in \mathcal{K}(\mathcal{D})} \mathcal{D}'$. By continuity of f_σ, we have $f_\sigma(\mathcal{D}) = \bigsqcup_{\mathcal{D}' \in \mathcal{K}(\mathcal{D})} f_\sigma(\mathcal{D}')$. Therefore, by Lemma 36 we have:

$$\{\{\mathcal{D}\}\} = \bigcup_{\mathcal{D}' \in \mathcal{K}(\mathcal{D})} \{\{\mathcal{D}'\}\} = \bigcup_{\mathcal{D}' \in \mathcal{K}(\mathcal{D})} \{\{f_\sigma(\mathcal{D}')\}\} = \{\{\bigsqcup_{\mathcal{D}' \in \mathcal{K}(\mathcal{D})} f_\sigma(\mathcal{D}')\}\} = \{\{f_\sigma(\mathcal{D})\}\}.$$ ◀

▶ **Corollary 38.** *Let* $\mathcal{D} \in \mathsf{wpPLL}^\infty$. *Then* $\{\{\mathcal{D}\}\} \neq \varnothing$.

Proof. If $\mathcal{D} \in \mathsf{wpPLL}^\infty$ is a cut-free coderivation, then weak-progressing ensures the existence of a bar $\mathcal{V}$ containing conclusions of rules in $\{\mathsf{ax}, 1, \mathsf{c!p}\}$. By weak König's lemma, $\lfloor \mathcal{D} \rfloor_\mathcal{V}$ is finite. Then, we prove by induction on $\lfloor \mathcal{D} \rfloor_\mathcal{V}$ that there is $n \geq 0$ such that $\{\{\lfloor \mathcal{D} \rfloor_\mathcal{V}\}\}_n \neq \varnothing$, so that we conclude $\varnothing \neq \{\{\lfloor \mathcal{D} \rfloor_\mathcal{V}\}\}_n \subseteq \{\{\mathcal{D}\}\}_n \subseteq \{\{\mathcal{D}\}\}$. As for the base case, notice that the interpretation of any coderivation ending with the c!p contains the element $(\vec{[\,]}, [\,])$, so it is never empty. The inductive steps are straightforward.

If $\mathcal{D}$ contains cut-rules, then $\{\{\mathcal{D}\}\} = \{\{f_\sigma(\mathcal{D})\}\}$ by Theorem 37. Since $f_\sigma(\mathcal{D})$ is cut-free, we conclude $\{\{\mathcal{D}\}\} \neq \varnothing$ by the above reasoning. ◀

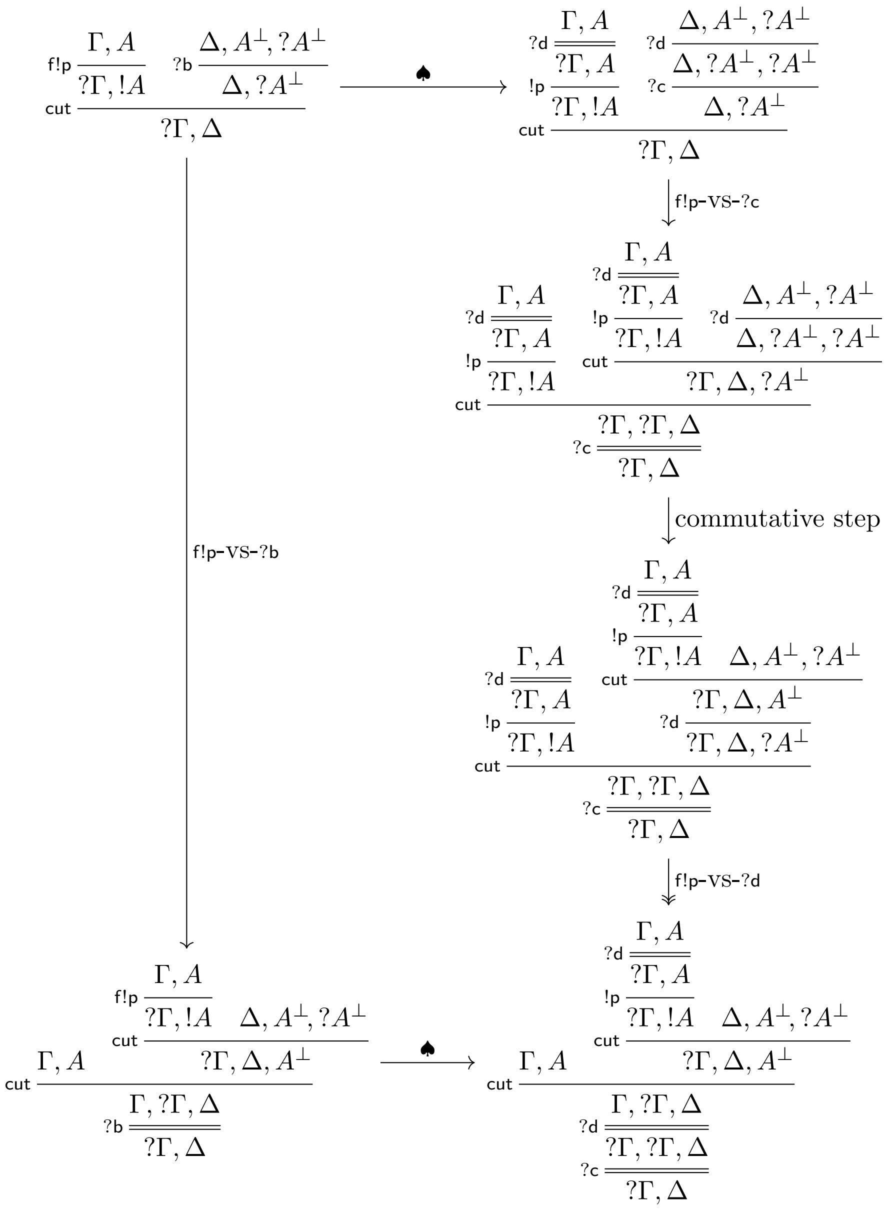


Figure 15 Commutation of the f!p-vs-?b step with $(\cdot)^{\spadesuit}$.

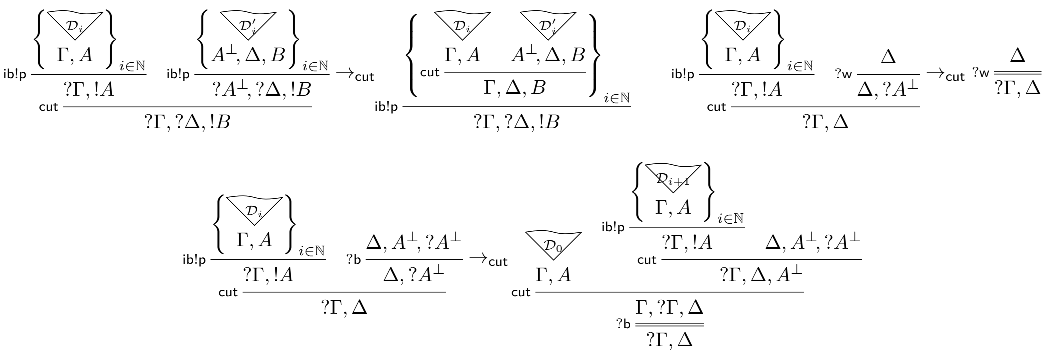


Figure 16 Exponential cut-elimination steps in nuPLL.