The Benefit of Being Non-Lazy in Probabilistic λ-calculus

Gianluca Curzi (Università di Torino)

Michele Pagani (Université de Paris)

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Introduction

- Program equivalence: contextual equivalence vs bisimilarity.

- *Applicative bisimilarity* [Abramsky 93].

$$\Lambda^{\text{cbn}} = \text{LTS}$$

- *Probabilistic applicative bisimilarity (PAB)* [Dal Lago et al. 13].

$$\Lambda_{\oplus}^{\text{cbn}} = \text{LMC}$$

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Full Abstraction = Soundness + Completeness.

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value = **function** (e.g. $\lambda x.M$)

		lazy		

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$\lambda x.(\lambda y.y)x \to_\beta \lambda x.x$

	lazy			*non-lazy*

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	lazy			*non-lazy*
	cbn	cbv		

$$(\lambda x.xx)(\mathbf{II}) \to_\beta (\lambda x.xx)\mathbf{I} \to_\beta \mathbf{II}$$

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	lazy			*non-lazy*
	cbn	cbv		
Soundness ($\sim \subseteq =_{\text{cxt}}$)	✓			
Completeness ($=_{\text{cxt}} \subseteq \sim$)	*X*			

[Dal Lago et al. 14]

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[Crubillé&Dal Lago 14]

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[Kasterovic&Pagani 19]

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via *Context Lemma* (no Howe's method)

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Full Abstraction = Soundness + Completeness.

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FA for $=_{\mathrm{PCoh}}$ [Ehrahrd et. al 11, Clairambault&Paquet 18]

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FA for $=_{\mathcal{PT}}$ [Leventis 18, Leventis&Pagani 2019]

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Roadmap

1. Probabilistic λ-calculus $\Lambda_\oplus$ and operational semantics $[\![\cdot]\!]$
2. Contextual equivalence ($=_{\mathrm{cxt}}$)
3. Labelled Markov Chain (LMC)
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5. Probabilistic Nakajima trees [Leventis 18]
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Probabilistic λ-calculus $\Lambda_\oplus$ and operational semantics $[\![\cdot]\!]$

- Probabilistic λ-calculus $\Lambda_\oplus$:

$$M := x \mid \lambda x.M \mid (MM) \mid M \oplus M$$
$$H := \lambda x_1 \ldots x_n.yM_1 \ldots M_m \qquad \text{(head nf)}$$

- *Big-step approximation* $\Downarrow \subseteq \Lambda_\oplus \times \mathfrak{D}(\mathrm{HEAD})$:

$$\frac{}{M \Downarrow \bot}\,s1 \qquad \frac{}{x \Downarrow x}\,s2 \qquad \frac{M \Downarrow \mathscr{D}}{\lambda x.M \Downarrow \lambda x.\mathscr{D}}\,s3 \qquad \frac{M \Downarrow \mathscr{D} \qquad N \Downarrow \mathscr{E}}{M \oplus N \Downarrow \frac{1}{2}\cdot\mathscr{D} + \frac{1}{2}\cdot\mathscr{E}}\,s4$$

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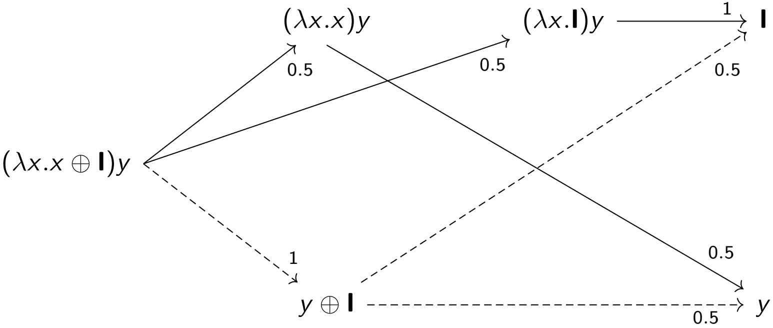
- Head reduction vs head spine reduction.

- Example:



- Theorem:

$$\forall M \in \Lambda_{\oplus},\ \forall H \in \mathrm{HEAD},\ \forall n \in \mathbb{N}: \quad \mathrm{Prob}^{n}_{head}[M, H] = \mathrm{Prob}^{n}_{spine}[M, H].$$

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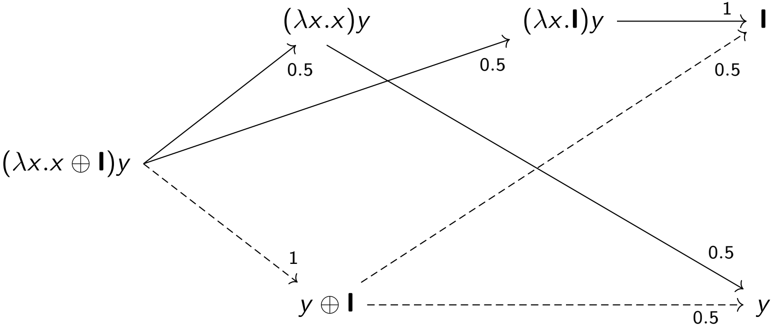

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$$M =_{\mathrm{cxt}} N \quad \text{iff} \quad \forall \mathcal{C} \quad \sum [\![\mathcal{C}[M]]\!] = \sum [\![\mathcal{C}[N]]\!].$$

- Example: $\lambda x.(x \oplus x) =_{\mathrm{cxt}} \lambda x.x$.

- Example: $\lambda z.z(\mathbf{\Omega} \oplus \mathbf{I}) \neq_{\mathrm{cxt}} \lambda z.(z\mathbf{\Omega} \oplus z\mathbf{I})$. If $\mathcal{C} \triangleq [\cdot]\mathbf{\Delta}$ then:

$$(\lambda z.z(\mathbf{\Omega} \oplus \mathbf{I}))\mathbf{\Delta} \xrightarrow[0.25]{}{}^{*} \mathbf{I}$$

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where $\mathbf{I} \triangleq \lambda x.x$, $\mathbf{\Delta} \triangleq \lambda x.xx$, and $\mathbf{\Omega} \triangleq \mathbf{\Delta\Delta}$.

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Labelled Markov Chain (LMC)

- $\Lambda_{\oplus}^{\text{head}}$ as a LMC:

$$M \xrightarrow[p]{\tau} \lambda x.H \qquad \cdots \qquad M \xrightarrow[p']{\tau} \lambda x.H'$$

$$\lambda x.H \xrightarrow[1]{M} H[M/x]$$

- Example: if $\mathbf{T} \triangleq \lambda xy.x$ and $\mathbf{F} \triangleq \lambda xy.y$ then:

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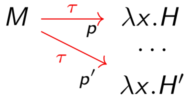

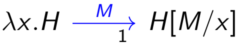

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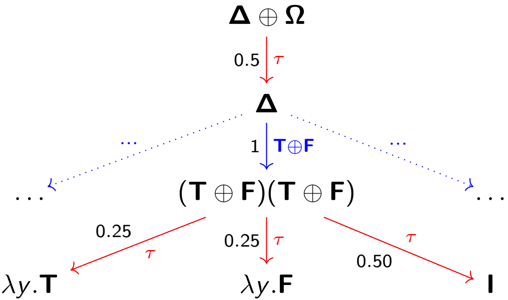

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Probabilistic applicative bisimilarity ($\sim$)

- *Probabilistic applicative bisimulation*: $\mathcal{R}$ = equivalence relation such that

$$\forall H \qquad \begin{array}{l} M \xrightarrow[p]{\tau} \\ \mathcal{R} \qquad\qquad \{H' \mid H' \,\mathcal{R}\, H\} \\ N \xrightarrow[p]{\tau} \end{array} \qquad\qquad \begin{array}{ccc} \lambda x.H & \xrightarrow[1]{M} & H[M/x] \\ \mathcal{R} & & \mathcal{R} \\ \lambda x.H' & \xrightarrow[1]{M} & H'[M/x] \end{array}$$

- Example: if fix $\triangleq (\lambda y.\mathbf{I} \oplus yy)(\lambda y.\mathbf{I} \oplus yy)$ then $\lambda x.x \oplus x \sim$ fix, since:

$$\lambda x.x \oplus x \xrightarrow[1]{\tau} \mathbf{I} \qquad\qquad \text{fix} \xrightarrow[1]{\tau} \mathbf{I}$$

so $\mathcal{R} \triangleq \{(\lambda x.x \oplus x, \text{fix}), (\text{fix}, \lambda x.x \oplus x)\} \cup \{(N, N) \mid N \in \Lambda^{\emptyset}_{\oplus}\}$ is bisimulation.

- Example: $\mathbf{\Delta} \not\sim \mathbf{I}$, since:

$$\mathbf{\Delta} \xrightarrow[1]{\Omega \oplus \mathbf{I}} (\Omega \oplus \mathbf{I})(\Omega \oplus \mathbf{I}) \xrightarrow[0.25]{\tau} \mathbf{I} \qquad\qquad \mathbf{I} \xrightarrow[1]{\Omega \oplus \mathbf{I}} (\Omega \oplus \mathbf{I}) \xrightarrow[0.50]{\tau} \mathbf{I}$$

Probabilistic applicative bisimilarity ($\sim$)

- Probabilistic applicative bisimilarity (PAB): $\sim$ = the "largest" bisimulation

$$\forall H \quad \begin{matrix} M \xrightarrow[p]{\tau} \\ \sim \\ N \xrightarrow[p]{\tau} \end{matrix} \{H' \mid H' \sim H\} \qquad \begin{matrix} \lambda x.H \xrightarrow[1]{M} H[M/x] \\ \sim \qquad\qquad \sim \\ \lambda x.H' \xrightarrow[1]{M} H'[M/x] \end{matrix}$$

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$$\mathbf{\Delta} \xrightarrow[1]{\Omega \oplus \mathbf{I}} (\Omega \oplus \mathbf{I})(\Omega \oplus \mathbf{I}) \xrightarrow[0.25]{\tau} \mathbf{I} \qquad\qquad \mathbf{I} \xrightarrow[1]{\Omega \oplus \mathbf{I}} (\Omega \oplus \mathbf{I}) \xrightarrow[0.50]{\tau} \mathbf{I}$$

Probabilistic applicative bisimilarity ($\sim$)

- Probabilistic applicative bisimilarity (PAB): $\sim$ = the "largest" bisimulation

$$\forall H \quad \begin{matrix} M \xrightarrow[p]{\tau} \\ \sim \\ N \xrightarrow[p]{\tau} \end{matrix} \{H' \mid H' \sim H\} \qquad\qquad \begin{matrix} \lambda x.H & \xrightarrow[1]{M} & H[M/x] \\ \sim & & \sim \\ \lambda x.H' & \xrightarrow[1]{M} & H'[M/x] \end{matrix}$$

- Example: if fix $\triangleq (\lambda y.\mathbf{I} \oplus yy)(\lambda y.\mathbf{I} \oplus yy)$ then $\lambda x.x \oplus x \sim$ fix, since:

$$\lambda x.x \oplus x \xrightarrow[1]{\tau} \mathbf{I} \qquad\qquad \text{fix} \xrightarrow[1]{\tau} \mathbf{I}$$

so $\mathcal{R} \triangleq \{(\lambda x.x \oplus x, \text{fix}), (\text{fix}, \lambda x.x \oplus x)\} \cup \{(N, N) \mid N \in \Lambda_{\oplus}^{\emptyset}\}$ is bisimulation.

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Roadmap

1. Probabilistic λ-calculus $\Lambda_\oplus$ and operational semantics $[\![\cdot]\!]$
2. Contextual equivalence ($=_{\mathrm{cxt}}$)
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Probabilistic Nakajima trees [Leventis 18]

- *Separation Theorem* [Leventis 18]: $M =_{cxt} N$ implies $\mathcal{PT}(M) = \mathcal{PT}(N)$.

- *Böhm tree* ($\mathcal{BT}$):

$\mathcal{BT}(\lambda x_1 \ldots x_n.yM_1 \ldots M_k) \triangleq$

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$\mathcal{PT}(M) \triangleq$

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$\ldots$

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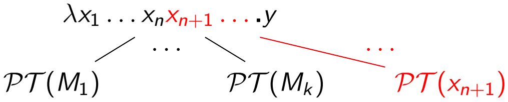

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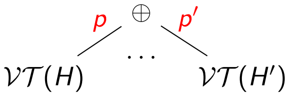

- Example: if $H \triangleq \lambda x.y(x \oplus y)$ then $\mathcal{PT}(H \oplus \mathbf{\Omega})$ is:

$$\oplus$$

$$0.5 \Big|$$

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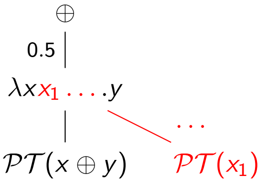

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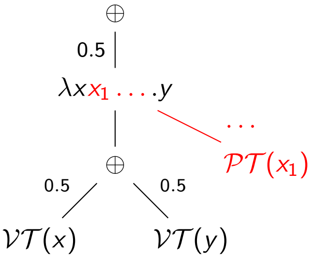

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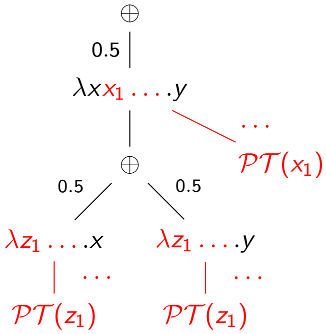

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Full abstraction

- Theorem (*Full abstraction*): Let $M, N \in \Lambda_\oplus$. We have $M \sim N$ iff $M =_{\mathrm{cxt}} N$.

- *Soundness* ($\sim \subseteq =_{\mathrm{cxt}}$): from $M \sim N$ we have

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  $\forall H$: $M \xrightarrow[p]{\tau} \{H' \mid \mathcal{VT}(H') = \mathcal{VT}(H)\}$, $M =_{\mathrm{cxt}} N$, $N \xrightarrow[q]{\tau} \{H' \mid \mathcal{VT}(H') = \mathcal{VT}(H)\}$

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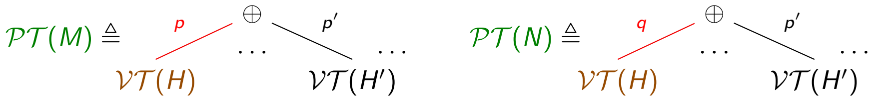

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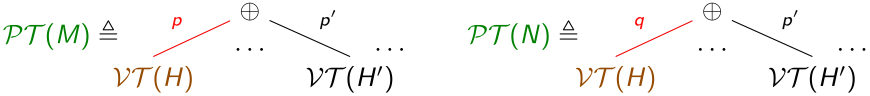

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Failure of full abstraction in the asymmetric case

- Theorem: Probabilistic applicative similarity ($\precsim$) is sound but not complete (hence fully abstract) with respect to contextual preorder ($\leq_{\mathrm{cxt}}$).

- **Counterexample**: similar to [Crubillé&Dal Lago 2014]

$$M \triangleq \lambda x.x(\mathbf{\Omega} \oplus \mathbf{I}) \qquad \text{vs} \qquad N \triangleq \lambda x.(x\mathbf{\Omega} \oplus x\mathbf{I})$$

- $\lambda x.x(\mathbf{\Omega} \oplus \mathbf{I}) \leq_{\mathrm{cxt}} \lambda x.(x\mathbf{\Omega} \oplus x\mathbf{I})$ by Context Lemma.

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Thank you!
Questions?

Probabilistic Applicative Similarity (PAS)

- *Probabilistic applicative simulation*: $\mathcal{R}$ = preorder relation such that

$$\begin{array}{ccc} M & \xrightarrow[p]{\tau} & X \subseteq \mathrm{HEAD} \\ \mathcal{R} & & \mathcal{R} \\ N & \xrightarrow[p' \geq p]{\tau} & \mathcal{R}(X) \end{array} \qquad \begin{array}{ccc} \lambda x.H & \xrightarrow[1]{M} & H[M/x] \\ \mathcal{R} & & \mathcal{R} \\ \lambda x.H' & \xrightarrow[1]{M} & H'[M/x] \end{array}$$

- Probabilistic applicative similarity (PAS): $\precsim$ = the "largest" simulation

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Counterexample to the asymmetric full abstraction