Linear Additives

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Introduction

- *Additive rules* of LL as type assignment rules:

$$\langle M, N \rangle : A \,\&\, B \qquad \pi_1 : A \,\&\, B \multimap A \qquad \pi_2 : A \,\&\, B \multimap B$$

- Variants of & to capture NP [Maurel 03, Matsuoka 04, Gaboardi et al. 08].

- Drawback: **exponential** blow up

$$\text{e.g.} \qquad (\lambda x.\langle x, x \rangle)M \quad \rightsquigarrow \quad \langle M, M \rangle$$

- *Lazy reduction* to "freeze" evaluation [Girard 96].

- This talk: new system LAM with *linear additive rules*, which allow **strong** linear normalization.

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- This talk: new system LAM with *linear additive rules*, which allow **strong** linear normalization.

Linear additives in a nutshell

- How to get rid of the exponential blow up?
- Avoid the increase of redexes $\rightsquigarrow$ **linear time normalization**.
- Avoid the increase of size $\rightsquigarrow$ **linear space normalization**.
- …but preserving Subject reduction.

$$\frac{x : A \vdash M_1 : B_1 \qquad x : A \vdash M_2 : B_2}{x : A \vdash \langle M_1, M_2 \rangle : B_1 \,\&\, B_2}\ \&\mathrm{R}$$

$$(\lambda x.\langle M_1, M_2\rangle)N \to \langle M_1, M_2\rangle[N/x] = \langle M_1[N/x], M_2[N/x]\rangle$$

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$$\frac{x_1 : A \vdash M_1 : B_1 \qquad x_2 : A \vdash M_2 : B_2}{x : A \vdash \texttt{copy}\ x\ \texttt{as}\ x_1, x_2\ \texttt{in}\ \langle M_1, M_2 \rangle : B_1 \mathbin{\&} B_2}\ \&\mathrm{R}$$

$$\texttt{copy}\ V\ \texttt{as}\ x_1, x_2\ \texttt{in}\ \langle M_1, M_2 \rangle \to \langle M_1[V/x_1], M_2[V/x_2] \rangle$$

V is a *value* (closed normal form).

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$$\mathtt{copy}^{U} V \ \mathtt{as} \ x_1, x_2 \ \mathtt{in} \ \langle M_1, M_2 \rangle \rightarrow \langle M_1[V/x_1], M_2[V/x_2] \rangle$$

U, V are *values* (closed normal forms).

A, B_1, B_2 are closed and $\forall$-*lazy types* (free from negative $\forall$).

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$$\dfrac{\vdash V : A \qquad \dfrac{x_1 : A \vdash N_1 : B_1 \qquad x_2 : A \vdash N_2 : B_2 \qquad \vdash U : A}{x : A \vdash \mathtt{copy}^U x \ \mathtt{as}\ x_1, x_2\ \mathtt{in}\ \langle N_1, N_2 \rangle : B_1 \,\&\, B_2}\,\&\mathrm{R}}{\vdash \mathtt{copy}^U V\ \mathtt{as}\ x_1, x_2\ \mathtt{in}\ \langle N_1, N_2 \rangle : B_1 \,\&\, B_2}\,\mathit{cut}$$

$$\downarrow$$

$$\dfrac{\vdash N_1[V/x_1] : B_1 \qquad \vdash N_2[V/x_2] : B_2}{\vdash \langle N_1[V/x_1], N_2[V/x_2] \rangle : B_1 \,\&\, B_2}\,\&\mathrm{R}\emptyset$$

Linearly Additive Multiplicative Type Assignment (LAM)

- System LAM = IMLL_2 + linear additive rules ("weaker" additives):

$$\frac{\Gamma, x_i : A_i \vdash M : C \qquad i \in \{1,2\}}{\Gamma, y : A_1 \& A_2 \vdash M[\pi_i(y)/x_i] : C}\ \&\mathrm{L}i \qquad \frac{\vdash M_1 : B_1 \qquad \vdash M_2 : B_2}{\vdash \langle M_1, M_2\rangle : B_1 \& B_2}\ \&\mathrm{R}\emptyset$$

$$\frac{x_1 : A \vdash M_1 : B_1 \qquad x_2 : A \vdash M_2 : B_2 \qquad \vdash V : A}{x : A \vdash \mathtt{copy}^{V} x\ \mathtt{as}\ x_1, x_2\ \mathtt{in}\ \langle M_1, M_2\rangle : B_1 \& B_2}\ \&\mathrm{R}$$

- Conditions
  - V is a **value**.
  - A, A_1, A_2, B_1, B_2 are **closed and $\forall$-lazy.**
  - *Closure*: if $\Gamma \vdash M : A$ and $FV(A) = \emptyset$ then $FV(\Gamma) = \emptyset$.

- Reduction rules:

$$
\begin{array}{rcll}
(\lambda x.M)N & \to & M[N/x] & \\
\pi_i\langle M_1, M_2\rangle & \to & M_i & i \in \{1,2\} \\
\texttt{copy}^U\, V \;\texttt{as}\; x_1, x_2 \;\texttt{in}\; \langle M_1, M_2\rangle & \to & \langle M_1[V/x_1], M_2[V/x_2]\rangle & U, V \text{ values}
\end{array}
$$

- **Restricted** cut-elimination rules:

$$
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$$

$$\rightsquigarrow_{\text{cut}}$$

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$$(\lambda x.M)N \to M[N/x]$$

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Basic properties of LAM

- Theorem ($\forall$-lazy cut-elimination): Let $\mathcal{D}$ be a derivation of a $\forall$-lazy type (no negative $\forall$). Then, $\mathcal{D}$ can be rewritten by the **restricted** cut-elimination rules to a cut-free derivation $\mathcal{D}^*$ in a **cubic** number of steps.

- Theorem (Subject reduction): If $\Gamma \vdash_{\mathsf{LAM}} M : A$ and $M \to N$, then:

  (i) $\mathrm{size}(N) < \mathrm{size}(M)$,

  (ii) $\Gamma \vdash_{\mathsf{LAM}} N : A$.

- Corollary (Strong linear normalization): If $\Gamma \vdash_{\mathsf{LAM}} M : A$ then M reduces to a normal form in at most $\mathrm{size}(M)$ steps.

On the expressiveness of IMLL_2

- IMLL_2 as type assignment for the linear λ-calculus:

$$\mathbf{I} : \mathbf{1} \qquad \lambda x.\texttt{let}\ x\ \texttt{be}\ \mathbf{I}\ \texttt{in}\ M : \mathbf{1} \multimap C$$

$$M \otimes N : A \otimes B \qquad \lambda x.\texttt{let}\ x\ \texttt{be}\ x_1 \otimes x_2\ \texttt{in}\ M : A \otimes B \multimap C$$

- Encoding **boolean circuits** in IMLL_2 [Mairson 03, Mairson&Terui 03].

$$\mathbf{B} \triangleq \forall \alpha.\alpha \multimap \alpha \multimap \alpha \otimes \alpha \qquad \underline{0} \triangleq \lambda x.\lambda y.x \otimes y \qquad \underline{1} \triangleq \lambda x.\lambda y.y \otimes x$$

- How to "linearly" express *fan-out*?

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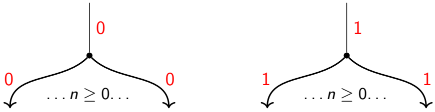

Linear erasure and duplication in IMLL_2

- Linear erasure by **consumption of data**:

$$\mathtt{E}_{\mathbf{B}} \triangleq \lambda z.\mathtt{let}\ z\mathbf{I}\mathbf{I}\ \mathtt{be}\ x, y\ \mathtt{in}\ (\mathtt{let}\ y\ \mathtt{be}\ \mathbf{I}\ \mathtt{in}\ x)$$

- Example:

$$\begin{aligned}\mathtt{E}_{\mathbf{B}}\,\underline{0} &\rightarrow \mathtt{let}\ (\lambda x.\lambda y.x \otimes y)\mathbf{I}\mathbf{I}\ \mathtt{be}\ x, y\ \mathtt{in}\ (\mathtt{let}\ y\ \mathtt{be}\ \mathbf{I}\ \mathtt{in}\ x) \\ &\rightarrow \mathtt{let}\ \mathbf{I} \otimes \mathbf{I}\ \mathtt{be}\ x, y\ \mathtt{in}\ (\mathtt{let}\ y\ \mathtt{be}\ \mathbf{I}\ \mathtt{in}\ x) \\ &\rightarrow \mathbf{I}\end{aligned}$$

- Linear duplication by **selection and erasure**:

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- **Generalizing** linear erasure and duplication to *closed* Π_1 *types* (no negative $\forall$):

$$\begin{array}{ccc} \mathbf{B} & \mapsto & \text{closed } \Pi_1 \\ = & & = \\ \text{boolean data type} & & \textbf{finite} \text{ data types} \end{array}$$

- Theorem [Mairson&Terui 03]:

$$\begin{array}{lcl} A \text{ closed } \Pi_1 & \mapsto & \mathtt{E}_A \\ A \text{ closed } \Pi_1 + \textbf{inhabited} & \mapsto & \mathtt{D}_A \end{array}$$

- Proposition [Curzi&Roversi 20]: $\mathrm{size}(\mathtt{D}_A)$ is **exponential** w.r.t $\mathrm{size}(A)$.
- **Sketch**: $\mathtt{D}_A$ stores any result of duplication $M \otimes M$, where M is a closed normal inhabitant of A.

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Translation into IMLL_2

- *Translation* $(\_)^\bullet : \mathsf{LAM} \to \mathsf{IMLL}_2$:

$$x_1 : A_1, \ldots, x_n : A_n \vdash_{\mathsf{LAM}} \overset{\mathcal{D}}{M} : B \quad \mapsto \quad x_1 : A_1^\bullet, \ldots, x_n : A_n^\bullet \vdash_{\mathsf{IMLL}_2} \mathcal{D}^\bullet : B^\bullet$$

- Intuitively:

$$\text{closed } \forall\text{-lazy} \quad (\&) \qquad \mapsto \qquad \text{closed } \Pi_1 \quad (\otimes)$$

$$\frac{\overset{\mathcal{D}_1}{x : A \vdash M : B} \quad \overset{\mathcal{D}_2}{y : A \vdash N : C} \quad \vdash V : A}{x : A \vdash \mathtt{copy}^V x \mathtt{\ as\ } x, y \mathtt{\ in\ } \langle M, N \rangle : B \,\&\, C} \quad \mapsto \quad \mathtt{let\ D}_{A^\bullet}\, x \mathtt{\ be\ } x \otimes y \mathtt{\ in\ } \mathcal{D}_1^\bullet \otimes \mathcal{D}_2^\bullet$$

$$\frac{\overset{\mathcal{D}}{\Gamma, x : A \vdash M : C}}{\Gamma, y : A \,\&\, B \vdash M[\pi_1(y)/x] : C} \quad \mapsto \quad \mathtt{let\ } x \mathtt{\ be\ } x \otimes x' \mathtt{\ in\ } (\mathtt{let\ E}_{B^\bullet}\, x' \mathtt{\ be\ I\ in\ } \mathcal{D}^\bullet)$$

Basic properties of the translation

- Theorem [Soundness of $(\_)^\bullet$] If $\mathcal{D} \rightsquigarrow_{\text{cut}} \mathcal{D}'$ in LAM, then $\mathcal{D}^\bullet \to^*_{\beta\eta} \mathcal{D}'^\bullet$.

- **Sketch**. The cut-elimination step for:

$$\dfrac{\dfrac{\mathcal{D}}{\vdash V : A}\, X \qquad \dfrac{\overset{\mathcal{D}_1}{x_1 : A \vdash M_1 : B_1} \qquad \overset{\mathcal{D}_2}{x_2 : A \vdash M_2 : B_2} \qquad \vdash U : A}{x : A \vdash \mathtt{copy}^U x \;\mathtt{as}\; x_1, x_2 \;\mathtt{in}\; \langle M_1, M_2 \rangle : B_1 \,\&\, B_2}\,\&R}{\vdash \mathtt{copy}^U V \;\mathtt{as}\; x_1, x_2 \;\mathtt{in}\; \langle M_1, M_2 \rangle : B_1 \,\&\, B_2}\,cut$$

is represented in IMLL_2 by:

$$\mathtt{let}\ \mathtt{D}_A\, \mathcal{D}^\bullet\ \mathtt{be}\ x \otimes y\ \mathtt{in}\ \mathcal{D}_1^\bullet \otimes \mathcal{D}_2^\bullet \to^*_{\beta\eta} \mathcal{D}_1^\bullet[\mathcal{D}^\bullet/x] \otimes \mathcal{D}_1^\bullet[\mathcal{D}^\bullet/x]$$

- Theorem [Exponential compression] If $\mathcal{D}$ is a derivation of $\Gamma \vdash_{\mathsf{LAM}} M : A$, then $\mathrm{size}(\mathcal{D}^\bullet)$ can be **exponential** w.r.t $\mathrm{size}(\mathcal{D})$.

- **Sketch**. $\mathrm{size}(\mathtt{D}_A) \in \mathcal{O}(2^{\mathrm{size}(A)^2})$.

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- Linear additives based on the Linear Logic additive disjunction $\oplus$?

	additives	linear additives
conjunction	$\&$	$\wedge$
disjuncton	$\oplus$	$\vee$

- STA_+ [Ronchi&Gaboardi 08]:

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : A}{\Gamma \vdash M + N : A}\,\mathrm{sum} \qquad\qquad M \leftarrow M + N \rightarrow N$$

"non-deterministic linear additives" to capture NP **regardless** of the reduction strategy.

- Same approach to capture PP and BPP (work in progress)?

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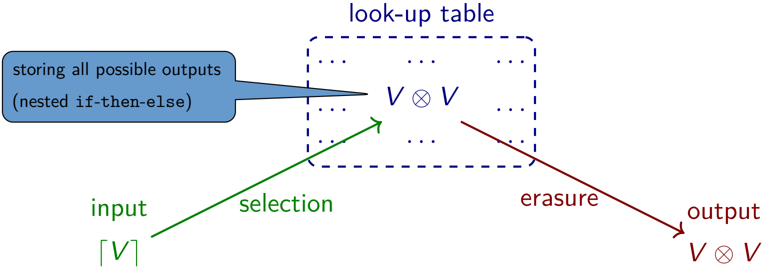
Thank you!
Questions?

Appendix

How duplicators work

Let A be a closed Π_1 type. The duplicator of A, written $\mathtt{D}_A$, implements two operations on a closed normal inhabitant V of A:

- **encode** V as a Boolean tuple $\lceil V \rceil$;
- **copy and decode** $\lceil V \rceil$ to obtain $V \otimes V$.



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